Applied Partial Differential Equations 1

Thomas P. Wihler

Department of Mathematics and Statistics
McGill University

Fall 2007

Contents

1

Introduction

In order to describe practical applications mathematically, suitable models are developed and applied. For example, many mechanical processes (such as, e.g., simple spring-mass systems) are based on Newton's law of motion: acceleration = applied force × mass. In many cases, these models can be expressed by means of differential equations (e.g., ordinary differential equations, partial differential equation, or integral equations). For instance, the position $x(t)$ at time t of a point of mass m satisfies the ordinary differential equation

$$F(t) = m\ddot{x}(t),$$

where $F(t)$ is the applied force at time t. The solution is easily obtained by twofold integration,

$$x(t) = \iint_t \ddot{x}(\tau)\,\mathrm{d}\tau = \iint_t F(\tau)\,\mathrm{d}\tau.$$

More generally, the solution of a differential equation can have different aspects (cf. also Figure 1.1):

- *analytical solution:* In some specific cases, it is possible to derive explicit solution formulas for differential equations. In general, this is, however, very difficult or not possible at all. For example, Lagrange showed that the (relatively simple) ordinary differential equation $\dot{x}(t) = x(t)^2 + t^2$ does not have a solution that can be expressed in terms of elementary functions.

- *properties of solutions:* Even if an explicit solution is not available, it is often possible to prove certain interesting properties about its behavior (including existence and uniqueness of solutions, local and global smoothness, location and character of singularities, monotonicity, preservation of physical properties, etc.). Such results are essential, for instance, in the design and analysis of suitable numerical solution methods.

- *numerical solution:* The design and investigation of numerical schemes for differential equations has become a major research area in science and related fields. Using mathematical tools, numerical methods can be tailored to specific problem classes and (also due to increasing computer power) highly accurate results can be obtained. Numerical schemes for differential equations which are often used (depending on the problem) include finite difference methods (FDM), finite volume methods (FVM), and finite element methods (FEM). In addition, many more numerical solution techniques are available.

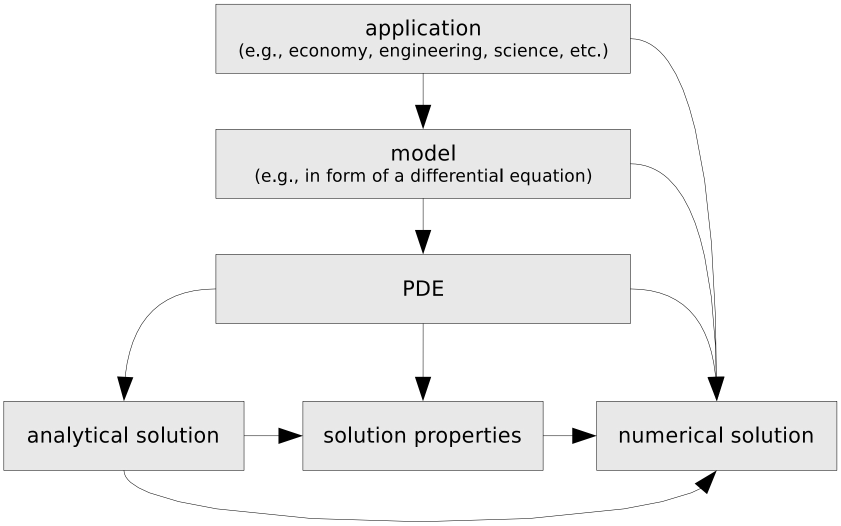

Figure 1.1: Applications and partial differential equations.

In this chapter we will review some important notation and definitions, and consider a number of examples of partial differential equations (PDE). Furthermore, we will introduce PDE in their general form, and will give some criteria for their classification. In addition, the question of well-posedness of a PDE problem will be addressed. Moreover, we shall briefly discuss the so-called separation of variables approach, which is one of the many possibilities to obtain PDE solutions.

1.1 Differential Operators

Let $m \in \mathbb{N}$, and $\Omega \subseteq \mathbb{R}^m$ an open set (called *domain*). The boundary of Ω will be denoted by $\partial\Omega$; cf. Figure 1.2. Then, for $1 \leq i \leq m$ and a (sufficiently) smooth function

$$u : \Omega \to \mathbb{R}, \qquad \boldsymbol{x} = (x_1, x_2, \dots x_m) \mapsto u(\boldsymbol{x}),$$

we define

$$\frac{\partial u}{\partial x_i} u(\boldsymbol{x}) = \lim_{h \to 0} \frac{u(\boldsymbol{x} + h\boldsymbol{e}_i) - u(\boldsymbol{x})}{h}$$

to be the *partial derivative* of u with respect to the co-ordinate direction x_i at $\boldsymbol{x}$ (provided that the two-sided limit exists). Here, $\boldsymbol{e}_i$ is the ith unit vector in $\mathbb{R}^m$, given by

$$[\boldsymbol{e}_i]_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

The partial derivative of order p (or pth partial derivative) of u, where $p \in \mathbb{N}$ with $p \geq 2$, in the x_j-direction is then defined iteratively by

$$\frac{\partial^p u}{\partial x_j^p} = \frac{\partial}{\partial x_j} \left(\frac{\partial^{p-1}}{\partial x_j^{p-1}} \right).$$

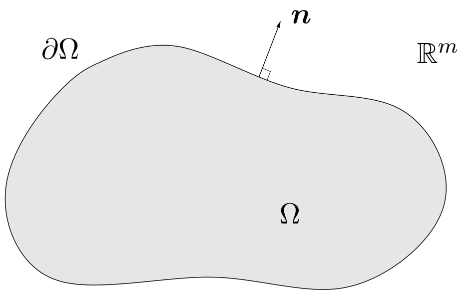

Figure 1.2: Domain Ω.

Furthermore, it is possible to consider partial derivatives in several co-ordinate directions. To this end, let $\boldsymbol{\alpha} = (i_1, i_2, \ldots, i_m) \in \mathbb{N}_0^m := (\mathbb{N} \cup \{0\})^m$, be a *multi-index*; its order is given by $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_m$. Then, the partial derivative of u corresponding to $\boldsymbol{\alpha}$ is defined by

$$\mathsf{D}^{\boldsymbol{\alpha}} u = \frac{\partial^{|\boldsymbol{\alpha}|} u}{\partial x_1^{i_1} \cdot \ldots \cdot \partial x_m^{i_m}}.$$

Furthermore, for $p \in \mathbb{N}_0$, we introduce the set

$$\mathsf{D}^p u = \{\mathsf{D}^{\boldsymbol{\alpha}} u : \, |\boldsymbol{\alpha}| = p\}.$$

The set of first order derivatives $\mathsf{D}^1 u$ is simply denoted by $\mathsf{D}u$.

Example 1.1 For $m = 3$ and $\boldsymbol{\alpha} = (2, 0, 3)$, the partial derivative corresponding to $\boldsymbol{\alpha}$ is given by

$$\mathsf{D}^{\boldsymbol{\alpha}} u = \frac{\partial^5 u}{\partial x_1^2 \partial x_3^3}.$$

Consider, for example, $u(x_1, x_2, x_3) = x_1 x_2 + e^{2x_1} x_3^4$. Then,

$$\mathsf{D}^{\boldsymbol{\alpha}} u = \frac{\partial^2}{\partial x_1^2} \left(\frac{\partial^3}{\partial x_3^3} \left(x_1 x_2 + e^{2x_1} x_3^4 \right) \right) = \frac{\partial^2}{\partial x_1^2} \left(24 e^{2x_1} x_3 \right) = 96 e^{2x_1} x_3.$$

◊

We note that, for vector-valued functions,

$$\boldsymbol{u} : \mathbb{R}^m \to \mathbb{R}^n, \qquad (x_1, \ldots, x_m) \mapsto \boldsymbol{u}(\boldsymbol{x}) = (u_1(\boldsymbol{x}), \ldots, u_n(\boldsymbol{x})), \tag{1.2}$$

where $m, n \in \mathbb{N}$, the above definitions are to be understood component-wise.

For a function as in (1.2), we shall consider three important differential operators that often appear in the context of PDE. First, the so-called *gradient* of $\boldsymbol{u}$ is defined as the following matrix:

$$\mathrm{grad}\boldsymbol{u} \equiv \nabla \boldsymbol{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_m} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_m} \end{bmatrix}.$$

This matrix is often referred to as the *Jacobian matrix* of $\boldsymbol{u}$. For $n = 1$, the gradient is a row vector,

$$\nabla u = \left[\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_m}\right].$$

Moreover, the *Laplacian* (or *Laplace operator*) of $\boldsymbol{u}$ is given by

$$\Delta \boldsymbol{u} = \sum_{i=1}^{m} \frac{\partial^2 \boldsymbol{u}}{\partial x_i^2}.$$

Finally, for $m = n$, the *divergence* of $\boldsymbol{u}$ is

$$\operatorname{div} \boldsymbol{u} \equiv \nabla \cdot \boldsymbol{u} = \sum_{i=1}^{m} \frac{\partial u_i}{\partial x_i}.$$

For $n = 1$, we note that there holds

$$\operatorname{div}(\nabla u) = \sum_{i=1}^{m} \frac{\partial \left(\frac{\partial u}{\partial x_i}\right)}{\partial x_i} = \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2} = \Delta u. \tag{1.3}$$

Furthermore, for a scalar function $u : \Omega \to \mathbb{R}$ and a vector-valued function $\boldsymbol{v} : \Omega \to \mathbb{R}^m$, there holds *Green's formula*,

$$\int_\Omega \nabla u(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} u(\boldsymbol{x})(\boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}))\,\mathrm{d}s_{\boldsymbol{x}} - \int_\Omega u(\boldsymbol{x}) \operatorname{div} \boldsymbol{v}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}, \tag{1.4}$$

where $\boldsymbol{n}(\boldsymbol{x})$ is the unit normal outward vector on $\partial\Omega$ at $\boldsymbol{x}$; cf. Figure 1.2. Here, we assume that Ω has a Lipschitz boundary[1], and that $u, \boldsymbol{v}$ are sufficiently smooth, so that the above integrals are well-defined. The identity (1.4) is an integration by parts formula in higher dimensions and constitutes one of the key tools in the analysis of PDE.

1.2 Examples of PDEs

We will now briefly consider a (very incomplete) list of well-known PDEs.

a) *Transport/Advection Equation:* Consider a domain $\Omega \subseteq \mathbb{R}^m$ (e.g., an ocean, lake, river, pipe, etc.), and a function $u = u(\boldsymbol{x}, t)$ depending on $\boldsymbol{x} \in \overline{\Omega}$ and time $t \geq 0$ (e.g., a concentration). Furthermore, let $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{x}, t)$ be a given velocity field (e.g., a current) and $f = f(\boldsymbol{x}, t)$ be a source or sink function. Then, the transport/advection equation is given by

$$\begin{array}{ccccc} \frac{\partial}{\partial t} u(\boldsymbol{x}, t) & = & \boldsymbol{b}(\boldsymbol{x}, t) \cdot \nabla u(\boldsymbol{x}, t) & + & f(\boldsymbol{x}, t) \\ \text{change in time} & = & \text{transported material} & + & \text{external source/sink} \end{array}$$

---

[1] We say that Ω has a Lipschitz boundary if $\partial\Omega$ can be parametrized locally by a Lipschitz continuous function.

b) *Heat equation:* Consider a source/sink $f = f(\boldsymbol{x}, t)$ and an unknown function $u = u(\boldsymbol{x}, t)$ (e.g., temperature) in a domain Ω. Then, the heat equation is

$$\frac{\partial}{\partial t} u(\boldsymbol{x}, t) = \Delta u(\boldsymbol{x}, t) + f(\boldsymbol{x}, t).$$

c) *Wave equation:* Consider, for example, a membrane Ω and a external force $f = f(\boldsymbol{x}, t)$. Then, the displacement $u = u(\boldsymbol{x}, t)$ of the membrane under the effect of the force f satisfies the wave equation,

$$\frac{\partial^2}{\partial t^2} u(\boldsymbol{x}, t) = \Delta u(\boldsymbol{x}, t) + f(\boldsymbol{x}, t).$$

Note that the heat equation and the wave differ just by one derivative of t. Nevertheless, the solution behavior of the two PDEs is completely different. For example, we shall see that the heat equation is dissipative (i.e. losing energy if $f \equiv 0$) while the wave equation is energy preserving (if $f \equiv 0$).

d) *Poisson equation:* Suppose that we consider a heat process that has been going on for a very long time and has become stationary. Mathematically, this means that the solution does not change in time, i.e. $\frac{\partial u}{\partial t} \equiv 0$. Recalling the heat equation (1.25), we thus obtain Poisson's equation,

$$-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}). \tag{1.5}$$

e) *First-order conservation laws:* These are PDEs for an unknown function $u = u(\boldsymbol{x}, t)$ of the form

$$\frac{\partial u}{\partial t} + \mathrm{div}_{\boldsymbol{x}} \boldsymbol{F}(u) = 0,$$

where $\boldsymbol{F}$ is a given vector-valued (typically nonlinear) function.

f) *Examples of systems of PDEs:*

  i) Linearized elasticity:

$$-\mu \Delta \boldsymbol{u} - (\mu + \lambda) \nabla (\mathrm{div} \boldsymbol{u}) = \boldsymbol{f}$$

  $\boldsymbol{u} =$ (vector-valued) displacement function
  $f =$ external force
  $\mu, \lambda =$ material coefficients.

  ii) (Incompressible) Navier-Stokes equations:

$$\begin{aligned} \boldsymbol{u}_t - \Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p &= \boldsymbol{f} \\ \mathrm{div} \boldsymbol{u} &= 0. \end{aligned}$$

  $\boldsymbol{u} =$ flow field,
  $p =$ pressure function.

1.3 Function Spaces

In order to classify functions according to their smoothness, we introduce a number of classical function spaces. Again, let Ω be an open set in $\mathbb{R}^m$. For simplicity, we restrict ourselves to scalar functions $u:\ \Omega \to \mathbb{R}$; the definitions for vector-valued functions are analogous.

1.3.1 Classical Spaces

By $\mathcal{C}^0(\overline{\Omega})$ we denote the space of all continuous functions on Ω. Furthermore, for $k \in \mathbb{N}$,

$$\mathcal{C}^k(\Omega) = \{u : \Omega \to \mathbb{R} :\ \mathsf{D}^{\boldsymbol{\alpha}} u \text{ exists and is continuous for all } \boldsymbol{\alpha} \in \mathbb{N}_0^m \text{ with } 0 \le |\boldsymbol{\alpha}| \le k\}$$

signifies the space of all k-times continuously differentiable functions on Ω. Moreover, consider

$$\begin{aligned}\mathcal{C}^k(\overline{\Omega}) = \big\{u : \mathcal{C}^k(\Omega) : \mathsf{D}^{\boldsymbol{\alpha}} u &\text{ is uniformly continuous on bounded subsets of } \Omega \\ &\text{for all } \boldsymbol{\alpha} \in \mathbb{N}_0^m \text{ with } 0 \le |\boldsymbol{\alpha}| \le k\big\},\end{aligned}$$

i.e., functions in $\mathcal{C}^k(\overline{\Omega})$ extend continuously to the boundary of Ω (wherever it exists). In addition, for $k \in \mathbb{N}_0$, we introduce the following norm on $\mathcal{C}^k(\overline{\Omega})$:

$$\|u\|_{\mathcal{C}^k(\overline{\Omega})} = \sum_{i=0}^{k} \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^m \\ |\boldsymbol{\alpha}| = i}} \sup_{\boldsymbol{x} \in \Omega} |\mathsf{D}^{\boldsymbol{\alpha}} u(\boldsymbol{x})|.$$

Moreover, we let $\mathcal{C}^\infty(\overline{\Omega})$ be the space of functions for which all partial derivatives of any order exist and are continuous:

$$\mathcal{C}^\infty(\overline{\Omega}) = \{u :\ \Omega \to \mathbb{R} : u \in \mathcal{C}^k(\overline{\Omega}) \text{ for all } k \in \mathbb{N}_0\}.$$

In many situations, it is interesting to consider functions with compact support. More precisely, for any of the function spaces above, we let

$$\mathcal{C}_c^k(\overline{\Omega}) = \{u \in \mathcal{C}^k(\overline{\Omega}) :\ \text{there exists a compact set } M \subset \Omega \text{ such that } u = 0 \text{ on } \Omega \setminus M\}.$$

Furthermore, for a bounded domain Ω, let us denote the subspace of all functions in $C^k(\overline{\Omega})$ that are zero along the boundary $\partial\Omega$ by $C_0^k(\overline{\Omega})$.

1.3.2 L^p-Spaces

In addition to the above classical spaces, we consider an further class of function spaces which appear often int he context of PDEs. Let us define, for $1 \le p < \infty$, the so-called L^p-norm on Ω

$$\|u\|_{p,\Omega} = \left(\int_\Omega |u(\boldsymbol{x})|^p \,\mathrm{d}\boldsymbol{x}\right)^{1/p}.$$

Then, the corresponding L^p-spaces are defined by

$$L^p(\Omega) = \{v :\ \|v\|_{p,\Omega} < \infty\}. \tag{1.6}$$

We remark that the precise definition of the L^p-norms and spaces is based on Lebesgue integration. In this way, highly nonsmooth functions are integrable and may thereby be elements of the L^p-spaces. For instance, the function

$$\chi:\ (0,1)\to\{0,1\},\qquad \chi(x)=\begin{cases}0 & \text{if } x\in\mathbb{Q}\\ 1 & \text{otherwise}\end{cases},$$

is not classically integrable (in the sense of Riemann integration), however, the Lebesgue integral

$$\int_0^1 |\chi(x)|^p \,\mathrm{d}x = 1$$

exists, and hence, $\chi\in L^p((0,1))$ for all $1\le p<\infty$.

We point out the following important properties.

Proposition 1.7 *Consider an open bounded domain* Ω. *Moreover, let* $p\in[1,\infty)$. *Then,* $\|\cdot\|_{p,\Omega}$ *is indeed a norm. In particular, there holds Minkowski's inequality (or triangle inequality),*

$$\|u+v\|_{p,\Omega}\le\|u\|_{p,\Omega}+\|v\|_{p,\Omega}$$

for all $u,v\in L^p(\Omega)$. *Moreover, consider* $q\in[1,\infty)$ *with*

$$\frac{1}{p}+\frac{1}{q}=1.$$

Then, Hölder's inequality,

$$\int_\Omega |u(\boldsymbol{x})v(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{x}\le\|u\|_{p,\Omega}\,\|v\|_{q,\Omega}\,, \tag{1.8}$$

is satisfied for any functions $u\in L^p(\Omega)$ *and* $v\in L^q(\Omega)$. *In particular, for* $u\in L^p(\Omega)$, $v\in L^{\frac{p}{p-1}}(\Omega)$, *we find* $uv\in L^1(\Omega)$.

Defining the integral product of two functions by

$$(u,v)_\Omega=\int_\Omega uv\,\mathrm{d}\boldsymbol{x}, \tag{1.9}$$

there holds $(u,u)=\|u\|_{2,\Omega}^2$ for any $u\in L^2(\Omega)$. Furthermore, applying (1.8) with $p=q=2$, we obtain

$$|(u,v)|\le\|u\|_{2,\Omega}\,\|v\|_{2,\Omega}\qquad\forall u,v\in L^2(\Omega).$$

Thus, Hölders inequality for $p=q=2$ can also be interpreted as a Cauchy-Schwarz inequality for the product (1.9).

Moreover, functions in L^p can be approximated by smooth functions (in the corresponding norm); see also Exercise 1.4.

Proposition 1.10 *The space* $C_0^\infty(\Omega)$ *is dense in* $L^p(\Omega)$ *for any* $p \in [1, \infty)$*, i.e., for any* $\varepsilon > 0$ *and* $\psi \in L^p(\Omega)$*, there exists* $\varphi \in C_0^\infty(\Omega)$ *such that*

$$\|\psi - \varphi\|_{p,\Omega} < \varepsilon.$$

An interesting consequence of the above result is, for example, the following:

Corollary 1.11 *Consider a function* $f \in L^2(\Omega)$ *which fulfills*

$$\int_\Omega f\varphi \, \mathrm{d}\boldsymbol{x} = 0 \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{1.12}$$

Then, there holds $f \equiv 0$*.*

Proof: Recalling Proposition 1.10, we can find $\widetilde{f} \in C_0^\infty(\Omega)$ such that $\left\|f - \widetilde{f}\right\|_{2,\Omega} < \varepsilon$ for any $\varepsilon > 0$. Hence,

$$\int_\Omega f^2 \, \mathrm{d}\boldsymbol{x} = \int_\Omega f\widetilde{f} \, \mathrm{d}\boldsymbol{x} + \int_\Omega f(f - \widetilde{f}) \, \mathrm{d}\boldsymbol{x}.$$

Furthermore, using (1.12), it follows that

$$\int_\Omega f^2 \, \mathrm{d}\boldsymbol{x} = \int_\Omega f(f - \widetilde{f}) \, \mathrm{d}\boldsymbol{x}.$$

Hence, with Hölder's inequality, we have

$$\|f\|_{2,\Omega}^2 = \int_\Omega f^2 \, \mathrm{d}\boldsymbol{x} \leq \|f\|_{2,\Omega} \left\|f - \widetilde{f}\right\|_{2,\Omega},$$

and therefore,

$$\|f\|_{2,\Omega} \leq \left\|f - \widetilde{f}\right\|_{2,\Omega} < \varepsilon \xrightarrow{\varepsilon \to 0} 0.$$

Thus, $\|f\|_{2,\Omega} = 0$. This implies that $f \equiv 0$. □

For functions which are zero along (at least parts of) the boundary $\partial\Omega$ of Ω, there holds the following result:

Proposition 1.13 *Consider an open bounded domain* $\Omega \subset \mathbb{R}^m$ *with a Lipschitz boundary* [2]*, and a function* $u \in \mathcal{C}^1(\overline{\Omega})$*. Moreover, assume that there exists a subset* $\Gamma \subseteq \partial\Omega$ *of the boundary of* Ω*, with* $\int_\Gamma \mathrm{d}s > 0$*, such that* $u = 0$ *on* Γ*. Then, there holds the* Poincaré-Friedrichs *inequality*

$$\|u\|_{2,\Omega} \leq C_{\Omega,\Gamma} \, \||\nabla u|\|_{2,\Omega}, \tag{1.14}$$

where $C_{\Omega,\Gamma} > 0$ *is a constant that depends only on* Ω *and* Γ*; in particular,* $C_{\Omega,\Gamma}$ *is independent of* u*.*

---

[2] The boundary of a domain is called Lipschitz, if it can be parametrized by a Lipschitz continuous function.

The above result is a special case of the first Poincaré-Friedrichs inequality. In fact, it holds for all $p \in [1, \infty)$ and is not limited to $p = 2$. Furthermore, it is applicable to more general classes of functions. We note, however, that it does not hold for functions which are nonzero on all of the boundary of Ω; the constant function $u \equiv 1$ illustrates this, for example. The second Poincaré-Friedrichs inequality refers to functions with zero-average on Ω. More precisely, for a function $u \in \mathcal{C}^1(\overline{\Omega})$ with $\int_\Omega u \,\mathrm{d}x = 0$, the bound (1.14) holds also true (with a different constant C_Ω); cf. Exercise 1.5.

Proof: [Proposition 1.13] We shall only prove the above result for the one-dimensional case, i.e. $\Omega = (a, b)$, $a < b$, is an open interval in $\mathbb{R}$.

Without loss of generality, we assume that $u(a) = 0$ and that $u \not\equiv 0$. Furthermore, let $\tau \in [a, b]$ be chosen such that $|u(\tau)| = \sup_{\boldsymbol{x} \in [a,b]} |u(x)|$. Then, there holds

$$u(\tau)^2 = \int_a^\tau \frac{\mathrm{d}}{\mathrm{d}x}\left(u(x)^2\right)\,\mathrm{d}x = 2\int_a^\tau u(x)u'(x)\,\mathrm{d}x. \tag{1.15}$$

Moreover, we note that, for any real numbers A, B, and $\delta > 0$, there holds the inequality

$$|AB| \leq \frac{\delta}{2}A^2 + \frac{1}{2\delta}B^2; \tag{1.16}$$

cf. Exercise 1.3. Hence, selecting $A = u(x)$, $B = u'(x)$, and $\delta = \frac{1}{2(\tau - a)}$ in (1.16), (1.15) becomes

$$\begin{aligned} u(\tau)^2 &\leq \frac{1}{2(\tau-a)}\int_a^\tau u(x)^2\,\mathrm{d}x + 2(\tau-a)\int_a^\tau u'(x)^2\,\mathrm{d}x \\ &\leq \frac{1}{2(\tau-a)}\int_a^\tau u(\tau)^2\,\mathrm{d}x + 2(b-a)\int_a^\tau u'(x)^2\,\mathrm{d}x \\ &\leq \frac{1}{2}u(\tau)^2 + 2(b-a)\int_a^b u'(x)^2\,\mathrm{d}x. \end{aligned}$$

Subtracting $\frac{1}{2}u(\tau)^2$ on either side of the above inequality, and multiplying by 2, results in

$$u(\tau)^2 \leq 4(b-a)\int_a^b u'(x)^2\,\mathrm{d}x = 4(b-a)\left\|u'\right\|_{2,[a,b]}^2.$$

Therefore, there holds

$$\|u\|_{2,[a,b]}^2 = \int_a^b u(x)^2\,\mathrm{d}x \leq (b-a)u(\tau)^2 \leq 4(b-a)^2\left\|u'\right\|_{2,[a,b]}^2.$$

This completes the proof. $\square$

An interesting question is whether the Poincaré-Friedrichs inequality in Proposition 1.13 holds also in the reverse direction. More precisely, is there a constant $\widetilde{C}_\Omega > 0$ such that

$$\|u'\|_{2,\Omega} \overset{?}{\leq} \widetilde{C}_\Omega \|u\|_{2,\Omega} \tag{1.17}$$

for all $u \in \mathcal{C}^1(\overline{\Omega})$? It can be seen quite easily that this is only true on finite-dimensional spaces (note that $\mathcal{C}^1(\overline{\Omega})$ is infinitely-dimensional); cf. Exercise 1.6. For example, it can be proved that there exists a constant $c_\Omega > 0$ such that

$$\|q'\|_{2,\Omega} \leq \widetilde{C}_\Omega r^2 \|q\|_{2,\Omega} \qquad \forall q \in \mathcal{P}_r(\Omega), \tag{1.18}$$

where $\mathcal{P}_r(\Omega) = \left\{q(x) = \sum_{i=0}^r a_i x^i : a_i \in \mathbb{R}, i = 0, 1, \ldots, r\right\}$ is the space of all polynomials of maximal degree p on a bounded open interval $\Omega \subset \mathbb{R}$. Results of this form are important, for example, in the analysis of high-order finite element methods for the numerical solution of PDE.

1.4 PDE in General Form and Classification

A partial differential equation (PDE) is an equation that is expressed in terms of an unknown function (of two or more variables) and some of its partial derivatives. The following definition introduces PDE in a very general form. More specific types of PDE will be discussed later on in this section.

Definition 1.19 *Let $\Omega \subseteq \mathbb{R}^m$ be an open domain, and $p \in \mathbb{N}$. Then, a (scalar)* partial differential equation (PDE) of order p *on Ω, in general form, is given by*

$$\mathsf{F}(u, \boldsymbol{x}) = 0, \tag{1.20}$$

where $u : \Omega \to \mathbb{R}$ is the unknown, *and*

$$\mathsf{F}(u, \boldsymbol{x}) = \Phi(\mathsf{D}^p u, \mathsf{D}^{p-1} u, \ldots, \mathsf{D}^2 u, \mathsf{D}u, u, \boldsymbol{x}). \tag{1.21}$$

Here, Φ is a (possibly nonlinear) function that depends on u and (at least some) of its partial derivatives up to and including order p, and the variable $\boldsymbol{x}$. Similarly, systems of PDEs are defined in terms of vector-valued quantities $\mathbf{F}$ and $\boldsymbol{\Phi}$.

The above definition is very general. In many cases, however, PDE have a quite specific form. This is important in practice, since the type of a PDE often provides some important information on its solution behavior.

Definition 1.22 *A partial differential equation* (1.20) *is*

- fully nonlinear *if (at least some of) the highest order (partial) derivatives of u (i.e. derivatives of order p) occur in a nonlinear form in the function Φ from* (1.21)*;*
- quasilinear *if the highest order derivatives of u occur only linearly (with coefficients that may depend on lower order derivatives of u and on $\boldsymbol{x}$) in Φ from* (1.21)*, i.e. the PDE is of the form*

$$\sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^m \\ |\boldsymbol{\alpha}| = p}} a_{\boldsymbol{\alpha}}(\mathsf{D}^{p-1} u, \ldots, \mathsf{D}^2 u, \mathsf{D}u, u, \boldsymbol{x}) \mathsf{D}^{\boldsymbol{\alpha}} u + \widetilde{\Phi}(\mathsf{D}^{p-1} u, \ldots, \mathsf{D}^2 u, \mathsf{D}u, u, \boldsymbol{x}) = 0;$$

- semilinear *if the highest order derivatives of* u *occur only linearly (with coefficients that only depend on* $\boldsymbol{x}$*) in* Φ *from* (1.21)*, i.e. the PDE is of the form*

$$\sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}_0^m \\ |\boldsymbol{\alpha}|=p}} a_{\boldsymbol{\alpha}}(\boldsymbol{x})\mathsf{D}^{\boldsymbol{\alpha}}u + \widehat{\Phi}(\mathsf{D}^{p-1}u,\dots,\mathsf{D}^2u,\mathsf{D}u,u,\boldsymbol{x}) = 0;$$

- linear *if* Φ *in* (1.21) *depends linearly on all but the last (i.e.* $\boldsymbol{x}$*) of its arguments, i.e. the PDE is of the form*

$$\sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}_0^m \\ |\boldsymbol{\alpha}|\le p}} a_{\boldsymbol{\alpha}}(\boldsymbol{x})\mathsf{D}^{\boldsymbol{\alpha}}u = f(\boldsymbol{x}).$$

Here, $\widetilde{\Phi}$*,* $\widehat{\Phi}$ *and* f *are suitable functions.*

Example 1.23 Consider a domain $\Omega \subseteq \mathbb{R}^m$.

1. The level set type equation
$$|\nabla u(\boldsymbol{x})| = f(\boldsymbol{x}),$$
where $|\cdot|$ is the usual Euclidean norm, for an unknown function $u : \Omega \to \mathbb{R}$ and a given function $f : \Omega \to \mathbb{R}$ is an example of a first-order fully nonlinear equation.

2. Let $d : \mathbb{R}_+ \times \Omega \to \mathbb{R}$, $c : \mathbb{R} \times \Omega \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ be sufficiently smooth functions. Then, the PDE given by
$$-\operatorname{div}\left(d(|\nabla u|, \boldsymbol{x})\nabla u\right) + c(u, \boldsymbol{x}) = f, \tag{1.24}$$
describes, for example, a diffusion-reaction type process of an unknown density or concentration $u : \Omega \to \mathbb{R}$; here, d and c are the diffusion and reaction coefficients, respectively, and f is an external source.

   If d depends on $|\nabla u|$ then (1.24) is a quasilinear second order PDE. For d independent of $|\nabla u|$, i.e. $d = d(\boldsymbol{x})$, (1.24) is semilinear (provided that c is nonlinear in u). If, in addition, c is linear with respect to u, then the PDE is linear. In particular, if $d(x) \equiv 1$ and $c \equiv 0$, then (1.24) is the Poisson equation (1.5).

3. Time-dependent problems can be cast into the form (1.20) by identifying the time variable t by an addition variable x_{m+1}. For example, the linear heat equation
$$\frac{\partial u}{\partial t} - \Delta u = f, \tag{1.25}$$
or the linear wave equation
$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f, \tag{1.26}$$
are second order linear PDE for an unknown function $u = u(\boldsymbol{x}, t) : \Omega_T \to \mathbb{R}$, on the spatial domain Ω and the time interval $(0, T)$, $T > 0$. Here, $\Omega_T = \Omega \times (0, T)$.

4. Definition 1.19 can be extended naturally to systems of PDE and vector-valued functions. As an example, we consider the Stokes equations for linear incompressible fluid flow,
$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= f \\ \operatorname{div} \boldsymbol{u} &= 0. \end{aligned}$$
This is a second order linear system of two PDE for two unknown functions $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ (flow vector field) and $p : \Omega \to \mathbb{R}$ (pressure).

♢

We shall now discuss some further classification for linear partial differential equations. For simplicity, we focus only on scalar second order PDE. They can be written as

$$-\sum_{i,j=1}^{m} a_{ij}(\boldsymbol{x}) \frac{\partial^2}{\partial x_i \partial x_j} u(\boldsymbol{x}) + \sum_{i=1}^{m} b_i(\boldsymbol{x}) \frac{\partial}{\partial x_i} u(\boldsymbol{x}) + c(\boldsymbol{x}) u(\boldsymbol{x}) = f(\boldsymbol{x}), \tag{1.27}$$

where $\{a_{ij}(\boldsymbol{x})\}_{i,j}$, $\{b_i(\boldsymbol{x})\}_i$ and $c(\boldsymbol{x})$ are coefficients, and $f : \Omega \to \mathbb{R}$ is a given function. In order to classify this PDE, we define the so-called *principle part* of (1.27) by

$$\mathsf{P}(u)(\boldsymbol{x}) = -\sum_{i,j=1}^{m} a_{ij}(\boldsymbol{x}) \frac{\partial^2}{\partial x_i \partial x_j} u(\boldsymbol{x}).$$

Then, for a given point $\boldsymbol{x}_0 \in \Omega$, we introduce the following quadratic form on $\mathbb{R}^m$:

$$Q_{\boldsymbol{x}_0} : \mathbb{R}^m \to \mathbb{R}, \qquad Q_{\boldsymbol{x}_0}(\xi_1, \xi_2, \ldots, \xi_m) = -\sum_{i,j=1}^{m} a_{ij}(\boldsymbol{x}_0) \xi_i \xi_j.$$

Writing this in matrix-vector form, we obtain

$$Q_{\boldsymbol{x}_0}(\xi_1, \xi_2, \ldots, \xi_m) = -\boldsymbol{\xi}^\top \boldsymbol{A}(\boldsymbol{x}_0) \boldsymbol{\xi},$$

where

$$\boldsymbol{\xi} = [\xi_1, \xi_2, \ldots, \xi_m]^\top,$$

and

$$\boldsymbol{A}(\boldsymbol{x}_0) = \begin{bmatrix} a_{11}(\boldsymbol{x}_0) & a_{12}(\boldsymbol{x}_0) & \cdots & a_{1m}(\boldsymbol{x}_0) \\ a_{21}(\boldsymbol{x}_0) & a_{22}(\boldsymbol{x}_0) & \cdots & a_{2m}(\boldsymbol{x}_0) \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}(\boldsymbol{x}_0) & a_{m2}(\boldsymbol{x}_0) & \cdots & a_{mm}(\boldsymbol{x}_0) \end{bmatrix}. \tag{1.28}$$

Note that, for sufficiently smooth u, there holds

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_j \partial x_i}, \qquad 1 \le i, j \le m.$$

Thus, we will always suppose that the matrix $\boldsymbol{A}(\boldsymbol{x}_0)$ is *symmetric*. Consequently, its eigenvalues are real. The type of the PDE (1.27) will now be defined in dependence on these eigenvalues. More precisely, we have the following definition.

Definition 1.29 *If* $\boldsymbol{A}(\boldsymbol{x}_0)$ *in* (1.28) *is*

- *strictly definite, i.e. all of its eigenvalues have the same sign and are nonzero, then* (1.27) *is called* elliptic *at* $\boldsymbol{x}_0$*;*
- *singular such that there exists one zero eigenvalue and all the other eigenvalues have the same sign, and*

  $$\operatorname{rank}\left[\boldsymbol{A}(\boldsymbol{x}_0), [b_1(\boldsymbol{x}_0), \ldots, b_m(\boldsymbol{x}_0)]^\top\right] = m,$$

  then (1.27) *is called* parabolic *at* $\boldsymbol{x}_0$*;*
- *indefinite such that all but one of its eigenvalues have the same sign and one has the opposite sign (and are nonzero), then* (1.27) *is called* hyperbolic *at* $\boldsymbol{x}_0$.

Let us illustrate the above definition in the following examples.

Example 1.30

1. For the Poisson equation (1.5), the matrix $\boldsymbol{A}$ in (1.28) is, for any point $\boldsymbol{x}_0$, the identity matrix in $\mathbb{R}^{m\times m}$. This implies that all eigenvalues are 1, and hence, (1.5) is elliptic.
2. The heat equation (1.25) is an example of a parabolic equation. In fact, identifying t with an additional x-variable, $x_{m+1} = t$, we see that

   $$\boldsymbol{A} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 0 \end{bmatrix} \in \mathbb{R}^{(m+1)\times(m+1)},$$

   for any $\boldsymbol{x}_0$, and $\boldsymbol{A}$ has one zero eigenvalue. Moreover, $\boldsymbol{b} = [0, \ldots, 0, -1]^\top$. This implies that $\operatorname{rank}[\boldsymbol{A}, \boldsymbol{b}] = m + 1$.
3. For the wave equation (1.26), we proceed in a similar way as for heat equation, and obtain

   $$\boldsymbol{A} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{bmatrix} \in \mathbb{R}^{(m+1)\times(m+1)}.$$

   Wee see that this PDE is hyperbolic.

◊

1.5 Well-Posedness

Partial differential equations typically appear in combination with additional conditions. Most often they are necessary in order to guarantee, for example, the uniqueness of a solution of the corresponding problem. For instance, the equation

$$-\Delta u(\boldsymbol{x}) + u(\boldsymbol{x}) = 1, \tag{1.31}$$

on the unit square $\Omega = [0,1]^2 \subset \mathbb{R}^2$, does not have a unique solution. Indeed,

$$u(x_1, x_2) = 1 + \alpha e^{\frac{1}{\sqrt{2}}(x_1+x_2)}$$

solves the problem for any constant $\alpha \in \mathbb{R}$. By prescribing, however, the solution of the problem on the boundary $\partial\Omega$, the problem becomes uniquely solvable. For example, imposing

$$u(\boldsymbol{x}) = 1 \qquad \text{on } \partial\Omega, \tag{1.32}$$

leads to a unique solution of (1.31), given by $u(\boldsymbol{x}) = 1$. Conditions of the form (1.32) are called *Dirichlet boundary conditions*. Alternatively, a so-called *Neumann boundary condition* could be given,

$$\nabla u(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) = 0 \qquad \text{on } \partial\Omega,$$

where $\boldsymbol{n}$ is the unit outward vector on $\partial\Omega$. Again, this implies a unique solution $u(\boldsymbol{x}) = 1$ (we emphasize, however, that posing Neumann conditions does not always lead to unique solutions).

For time-dependent problems, the uniqueness of the solution requires, in addition to the boundary conditions (BC), one or more so-called *initial conditions (IC)*. Such conditions prescribe, for example, the solution of the PDE (or some of its partial derivatives) at a certain initial time t_0. For example, for the heat equation, in addition to a boundary condition, an initial condition of the form

$$u(\boldsymbol{x}, t = t_0) = g(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in \Omega,$$

where g is a given function, could be given. For the wave equation, which is a second-order problem with respect to time, a second initial condition (e.g., on the initial velocity $\frac{\partial u}{\partial t}$) is necessary to ensure unique solutions.

The boundary and initial conditions, as well as further given information (such as for example the given function f in (1.27)) are generally refereed to as *data*. The data is typically chosen from a suitable data space $\mathcal{X}_{\text{data}}$, and the solution of the corresponding PDE is sought for in an appropriate solution space $\mathcal{X}_{\text{solution}}$, cf. Figure 1.3. We suppose that $\mathcal{X}_{\text{data}}$and $\mathcal{X}_{\text{data}}$ are normed spaces, i.e., equipped with some suitable norms $\|\cdot\|_{\mathcal{X}_{\text{data}}}$ and $\|\cdot\|_{\mathcal{X}_{\text{solution}}}$, respectively.

Definition 1.33 *A PDE problem in the spaces* $\mathcal{X}_{\text{data}}, \mathcal{X}_{\text{solution}}$ *is* well-posed *if all of the following three conditions are satisfied:*

1. *for each data* $F \in \mathcal{X}_{\text{data}}$*, there exists a solution* $u \in \mathcal{X}_{\text{solution}}$ *in the given solution space* $\mathcal{X}_{\text{solution}}$*;*
2. *the solution* $u \in \mathcal{X}_{\text{solution}}$ *is unique;*

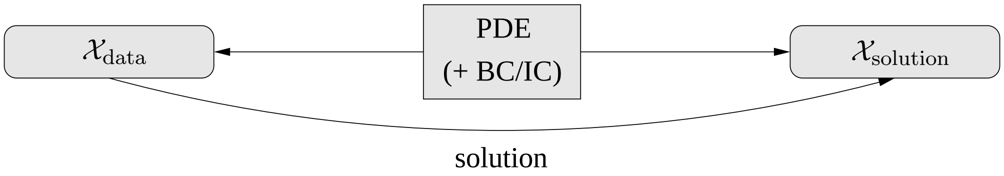

Figure 1.3: Solution process of PDEs: from data space to solution space.

3. *the solution $u \in \mathcal{X}_{\text{solution}}$ depends continuously on the data, i.e., for any given data $F \in \mathcal{X}_{\text{data}}$, the corresponding unique solution satisfies the* stability *bound*

$$\|u\|_{\text{solution}} \leq C\,\|F\|_{\text{data}}\,, \tag{1.34}$$

where $C > 0$ is a constant independent of the data F.

The following example is to illustrate this definition:

Example 1.35 Let $\Omega \subset \mathbb{R}^m$ be bounded and open. We assume that the boundary $\partial\Omega$ of Ω is split into two disjoint parts Γ_D and Γ_N: $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$. Consider Poisson's equation

$$-\Delta u = f \qquad \text{in } \Omega, \tag{1.36}$$

with (homogeneous) Dirichlet boundary conditions,

$$u = 0 \qquad \text{on } \Gamma_D, \tag{1.37}$$

and (homogeneous) Neumann boundary conditions,

$$\nabla u \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma_N.$$

Here, $f : \Omega \to \mathbb{R}$ is a given function (data), and $\boldsymbol{n}$ is the unit outward vector to $\partial\Omega$. We suppose that the boundary of Ω is smooth and $\int_{\Gamma_D} \mathrm{d}s > 0$, and that f belongs to the space

$$\mathcal{X}_{\text{data}} = \mathcal{C}^0(\overline{\Omega}).$$

Under these assumptions, the above problem has a unique solution in the space

$$\mathcal{X}_{\text{solution}} = \{u \in \mathcal{C}^2(\overline{\Omega}) :\ u = 0 \text{ on } \Gamma_D\};$$

this will be proved later. Here, we will focus on the stability estimate (1.34).

We equip the above spaces with the norms

$$\|f\|_{2,\Omega} = \left(\int_\Omega f(\boldsymbol{x})^2 \,\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{2}},$$

and

$$|\!|\!| u |\!|\!| = \left(\int_\Omega |\nabla u(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{2}}, \tag{1.38}$$

respectively; cf. Exercise 1.7. By definition, there holds

$$\|u\|^2 = \int_\Omega \nabla u \cdot \nabla u \,\mathrm{d}\boldsymbol{x}.$$

Integration by parts (Green's formula (1.4)), yields

$$\|u\|^2 = \int_{\partial\Omega} u(\nabla u\cdot\boldsymbol{n})\,\mathrm{d}s - \int_\Omega u\Delta u\,\mathrm{d}\boldsymbol{x} = \int_{\Gamma_D} u(\nabla u\cdot\boldsymbol{n})\,\mathrm{d}s + \int_{\Gamma_N} u(\nabla u\cdot\boldsymbol{n})\,\mathrm{d}s - \int_\Omega u\Delta u\,\mathrm{d}\boldsymbol{x}.$$

Noticing that $u = 0$ on Γ_D and $\nabla u \cdot \boldsymbol{n} = 0$ on Γ_N, and that $-\Delta u = f$, we obtain

$$\|u\|^2 = \int_\Omega f u \,\mathrm{d}\boldsymbol{x}.$$

Applying Hölder's inequality (1.8) with $p = q = 2$, results in

$$\|u\|^2 \le \int_\Omega |fu| \,\mathrm{d}\boldsymbol{x} \le \|f\|_{2,\Omega}\, \|u\|_{2,\Omega}\,.$$

Furthermore, the Poincaré inequality (1.14) implies that

$$\|u\|^2 \le C_\Omega \|f\|_{2,\Omega}\, \|u\|.$$

Dividing both sides of the above inequality by $\|u\|$, leads to

$$\|u\| \le C_\Omega \|f\|_{2,\Omega}\,. \tag{1.39}$$

This yields (1.34), and hence, the problem is well-posed in the corresponding spaces and norms. In particular, the solution is bounded by the corresponding data. ♢

1.6 Separation of Variables and Fourier Series

For the solution of PDE, many different approaches exist (power series, transformation techniques, integral representations, numerical methods, etc.). In this section, we shall focus on the so-called *separation of variables* method, which constitutes a very popular and relatively easily accessible way to solve (certain) PDEs problems. The basic idea is to assume that the unknown solution

$$u : \mathbb{R}^m \supseteq \Omega \to \mathbb{R}, \qquad \boldsymbol{x} = (x_1, x_2, \ldots, x_m) \mapsto u(\boldsymbol{x})$$

of a PDE can be split into m (additive or multiplicative) parts $U_i : \Omega \to \mathbb{R}$, $1 \le i \le m$, such that each of them depends on only one of the variables, i.e.

$$u(\boldsymbol{x}) = U_1(x_1) + U_2(x_2) + \ldots + U_m(x_m), \tag{1.40}$$

or

$$u(\boldsymbol{x}) = U_1(x_1)U_2(x_2)\ldots U_m(x_m). \tag{1.41}$$

Example 1.42 Let us consider the heat equation in 1-D. We are looking for a function $u = u(x,t) : \Omega \times (0,\infty) \to \mathbb{R}$ such that there holds

$$u_t - u_{xx} = 0 \qquad \text{in } \Omega = (0,1), \tag{1.43}$$

with boundary conditions

$$u(0,t) = u(1,t) = 0 \qquad \forall t > 0, \tag{1.44}$$

and initial condition

$$u(x,0) = \sin(\pi x) \qquad \forall x \in \Omega = (0,1). \tag{1.45}$$

Here, $(\cdot)_t$ and $(\cdot)_{xx}$ are used to mean $\frac{\partial}{\partial t}$ and $\frac{\partial^2}{\partial x^2}$, respectively. In order to solve this problem, we shall apply a separation of variables approach of the form (1.41), i.e.

$$u(x,t) = w(x)\varphi(t), \tag{1.46}$$

where the functions w and φ are to be found. Inserting (1.46) into (1.43), results in

$$w(x)\dot{\varphi}(t) - w''(x)\varphi(t) = 0,$$

where $(\cdot)'$ and $\dot{(\cdot)}$ denote the partial derivatives with respect to x and t, respectively. Hence, we obtain

$$\frac{\dot{\varphi}(t)}{\varphi(t)} = \frac{w''(x)}{w(x)}.$$

Note that the left-hand side of the above equation depends only on t while the right-hand side depends only on x. Therefore, both sides must be constant, i.e. there exists $\mu \in \mathbb{R}$ such that

$$\frac{\dot{\varphi}(t)}{\varphi(t)} = \frac{w''(x)}{w(x)} = \mu \qquad \forall (x,t) \in (0,1) \times (0,\infty).$$

From this, it follows that

$$\varphi(t) = Ce^{\mu t}, \tag{1.47}$$

and that

$$w(x) = D_1 e^{\lambda x} + D_2 e^{-\lambda x},$$

for some suitable constants $C \in \mathbb{R}$, and $D_1, D_2, \lambda \in \mathbb{C}$, with $\lambda^2 = \mu$. Furthermore, the boundary conditions (1.44) for u imply that

$$w(0) = w(1) = 0,$$

i.e. w fulfills the same boundary conditions like $u(x,t)$. Thus,

$$w(x) = D\sin(k\pi x), \tag{1.48}$$

for some constants $D \in \mathbb{R}$ and $k \in \mathbb{N}$, i.e. $D_1 = -D_2$, $\lambda = ik\pi$, and $\mu = -k^2\pi^2$. Combining (1.47)–(1.48), we obtain

$$u(x,t) = \widetilde{C}e^{-k^2\pi^2 t}\sin(k\pi x),$$

for some constant $\widetilde{C} \in \mathbb{R}$. Recalling the initial condition (1.45), we have

$$u(x,0) = \widetilde{C} \sin(k\pi x) = \sin(\pi x),$$

and therefore $\widetilde{C} = 1$ and $k = 1$. Thus,

$$u(x,t) = e^{-\pi^2 t} \sin(\pi x)$$

solves the problem (1.43)–(1.45). $\diamondsuit$

In many situations, a simple approach as in (1.41), for example, does not work, particularly if the solution of a problem does not have a corresponding form. The idea can, however, be generalized quite easily. In fact, instead of writing the solution $u(\boldsymbol{x})$ in (1.41) as a simple product, we represent it as a sum of products of the form (1.41):

$$u(\boldsymbol{x}) = \sum_k U_{k,1}(x_1) U_{k,2}(x_2) \dots U_{k,m}(x_m), \tag{1.49}$$

where the functions $\{U_{k,l}\}_{k,1 \leq l \leq m}$ are to be found. Often (but not always) in practice, some or all of the $U_{k,l}$ are written in terms of trigonometric functions, i.e. the sum (1.49) is a Fourier type series (or similar). We will illustrate this in the following example.

Example 1.50 We consider again the heat equation (1.43) on $\Omega = (0,1)$. Here, however, the boundary conditions are given by

$$u(0,t) = u_x(1,t) = 0 \qquad \forall t > 0, \tag{1.51}$$

and the initial condition is prescribed by

$$u(x,0) = x(x-1)^2. \tag{1.52}$$

We write the solution as

$$u(x,t) = \sum_{k=0}^{\infty} c_k(t) \sin\left(\frac{(2k+1)\pi}{2} x\right). \tag{1.53}$$

Although an alternative representation formula might lead to the solution of the problem, we note that the above choice is particularly advantageous. The reason for this is that, for u as in (1.53), both of the boundary conditions in (1.51) are satisfied for any coefficients $\{c_k(t)\}_{k \in \mathbb{N}_0}$, $t > 0$. Inserting (1.53) into the PDE (1.43), gives

$$\sum_{k=0}^{\infty} \left(\dot{c}_k(t) + \frac{(2k+1)^2 \pi^2}{4} c_k(t) \right) \sin\left(\frac{(2k+1)\pi}{2} x\right) = 0.$$

Since this equation holds for any $x \in (0,1)$, it follows that

$$\dot{c}_k(t) + \frac{(2k+1)^2 \pi^2}{4} c_k(t) = 0$$

for any $k \in \mathbb{N}_0$. Hence,

$$c_k(t) = C_k e^{-\frac{(2k+1)^2\pi^2}{4}t},$$

for suitable constants $\{C_k\}_{k\in\mathbb{N}_0}$, and therefore,

$$u(x,t) = \sum_{k=0}^{\infty} C_k e^{-\frac{(2k+1)^2\pi^2}{4}t} \sin\left(\frac{(2k+1)\pi}{2}x\right). \tag{1.54}$$

In order to find the coefficients C_k, we expand the initial condition (1.52) into a series of the same form as in (1.53) (this is possible since the initial condition is *compatible* with the boundary conditions), i.e.

$$x(x-1)^2 = \sum_{k=0}^{\infty} D_k \sin\left(\frac{(2k+1)\pi}{2}x\right), \tag{1.55}$$

for some constants $\{D_k\}_{k\in\mathbb{N}_0}$. Then, combining (1.54) and (1.55), leads to

$$\sum_{k=0}^{\infty} C_k \sin\left(\frac{(2k+1)\pi}{2}x\right) = u(x,0) = x(x-1)^2 = \sum_{k=0}^{\infty} D_k \sin\left(\frac{(2k+1)\pi}{2}x\right)$$

for any $x \in (0,1)$, and thus,

$$C_k = D_k \qquad \forall k \in \mathbb{N}_0. \tag{1.56}$$

The coefficients D_k in (1.55) are calculated by multiplying (1.55) by $\sin\left(\frac{(2l+1)\pi}{2}x\right)$ and integrating from 0 to 1. Formally, this yields

$$\int_0^1 x(x-1)^2 \sin\left(\frac{(2l+1)\pi}{2}x\right)\,\mathrm{d}x = \sum_{k=0}^{\infty} D_k \int_0^1 \sin\left(\frac{(2k+1)\pi}{2}x\right)\sin\left(\frac{(2l+1)\pi}{2}x\right)\,\mathrm{d}x.$$

Then, using that

$$\int_0^1 \sin\left(\frac{(2k+1)\pi}{2}x\right)\sin\left(\frac{(2l+1)\pi}{2}x\right)\,\mathrm{d}x = \begin{cases} \frac{1}{2} & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases},$$

implies that

$$\int_0^1 x(x-1)^2 \sin\left(\frac{(2k+1)\pi}{2}x\right)\,\mathrm{d}x = \frac{1}{2}D_k,$$

and therefore, we obtain

$$D_k = 2\int_0^1 x(x-1)^2 \sin\left(\frac{(2k+1)\pi}{2}x\right)\,\mathrm{d}x = \frac{32\left(4k\pi + 2\pi - 6(-1)^k\right)}{\pi^4(16k^4 + 32k^3 + 24k^2 + 8k + 1)}.$$

Hence, referring to (1.54) and (1.56), we see that

$$u(x,t) = \sum_{k=0}^{\infty} \frac{32\left(4k\pi + 2\pi - 6(-1)^k\right)}{\pi^4(16k^4 + 32k^3 + 24k^2 + 8k + 1)} e^{-\frac{(2k+1)^2\pi^2}{4}t} \sin\left(\frac{(2k+1)\pi}{2}x\right). \tag{1.57}$$

In practice, the above sum could be evaluated by truncation, for example. More precisely, for $k_0 \in \mathbb{N}$, consider the approximate sum

$$S_{k_0}(x,t) = \sum_{k=0}^{k_0} \frac{32\left(4k\pi + 2\pi - 6(-1)^k\right)}{\pi^4(16k^4+32k^3+24k^2+8k+1)} e^{-\frac{(2k+1)^2\pi^2}{4}t} \sin\left(\frac{(2k+1)\pi}{2}x\right).$$

In applications, it is often desirable to guarantee that an approximation is sufficiently accurate. More precisely, given a certain tolerance $\tau > 0$, we would like to find $k_0 \in \mathbb{N}$ such that

$$|u(x,t) - S_{k_0}(x,t)| \leq \tau, \tag{1.58}$$

for any $(x,t) \in (0,1) \times (0,\infty)$. In order to estimate the above error, we shall use that

$$\left|\sin\left(\frac{(2k+1)\pi}{2}x\right)\right| \leq 1 \qquad \forall x \in (0,1),$$

and that

$$\left|e^{-\frac{(2k+1)^2\pi^2}{4}t}\right| \leq e^{-\frac{(2k_0+1)^2\pi^2}{4}t} \qquad \forall t \geq 0, \forall k \geq k_0.$$

Furthermore, we notice that, for $k \geq 1$,

$$\begin{aligned}\left|\frac{32\left(4k\pi + 2\pi - 6(-1)^k\right)}{\pi^4(16k^4+32k^3+24k^2+8k+1)}\right| &\leq \frac{32\left(4k\pi + 2\pi + 6\right)}{\pi^4(16k^4+32k^3+24k^2+8k+1)} \\ &\leq \frac{32\left(4k\pi + 8\pi\right)}{\pi^4(16k^4+32k^3)} \leq \frac{8\pi\left(k+2\right)}{\pi^4(k^4+2k^3)} = \frac{8}{\pi^3k^3}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&|u(x,t) - S_{k_0}(x,t)| \\ &\quad\leq \sum_{k=k_0+1}^{\infty} \left|\frac{32\left(4k\pi + 2\pi - 6(-1)^k\right)}{\pi^4(16k^4+32k^3+24k^2+8k+1)}\right| \left|e^{-\frac{(2k+1)^2\pi^2}{4}t}\right| \left|\sin\left(\frac{(2k+1)\pi}{2}x\right)\right| \\ &\quad\leq \frac{8}{\pi^3} e^{-\frac{(2k_0+1)^2\pi^2}{4}t} \sum_{k=k_0+1}^{\infty} \frac{1}{k^3}.\end{aligned}$$

Using that $\sum_{k=1}^{\infty} \frac{1}{k^3} \leq 1.3$, results in

$$\sum_{k=k_0+1}^{\infty} \frac{1}{k^3} = \sum_{k=1}^{\infty} \frac{1}{k^3} - \sum_{k=1}^{k_0} \frac{1}{k^3} \leq 1.3 - \sum_{k=1}^{k_0} \frac{1}{k^3}.$$

Hence, in order to guarantee that (1.58) is satisfied, we choose $k_0 = k_0(t)$ large enough so that

$$\frac{8}{\pi^3} e^{-\frac{(2k_0+1)^2\pi^2}{4}t} \left(1.3 - \sum_{k=1}^{k_0} \frac{1}{k^3}\right) \leq \tau.$$

For example, for $t = \frac{1}{4}$ and $\tau = 10^{-16}$, we need $k_0 \geq 4$. Of course, the above calculations require implicitly that the series solution (1.57) represents indeed the solution of the original PDE problem. ♢

1.7 Exercises

1.1. Consider an open bounded interval $\Omega = (a,b) \subset \mathbb{R}$, $a < b$, and $f \in C^k(\overline{\Omega})$, for some $k \in \mathbb{N}_0$. Find a solution formula for Poisson's equation in 1-D with zero Dirichlet boundary conditions:

$$-u''(x) = f(x), \qquad u(a) = u(b) = 0.$$

To which function space does the solution u belong to?

1.2. Consider the function $u(x) = x^\beta$ on the domain $\Omega = [0,1]$, and a real number $1 \leq p < \infty$. For which values of β is $\|u\|_{p,\Omega}$ bounded?

1.3. Prove (1.16). Furthermore, use this result to prove Hölder's inequality (1.8) for $p = q = 2$ (Cauchy-Schwarz inequality for integrals).

1.4. Suppose Ω has a sufficiently smooth boundary. Prove that $C^0(\Omega)$ is dense in $L^p(\Omega)$ for $1 \leq p < \infty$.

1.5. Prove the 2nd Poincaré inequality in 1-D. More precisely, for a continuously differentiable function $u : [a,b] \to \mathbb{R}$, $a < b$, show that there exists a constant $\widetilde{C}_\Omega > 0$ (which only depends on a, b) such that there holds

$$\int_a^b (u(x) - \overline{u})^2 \,\mathrm{d}x \leq \widetilde{C}_\Omega \int_a^b u'(x)^2 \,\mathrm{d}x.$$

Here,

$$\overline{u} = \frac{1}{b-a} \int_a^b u(x) \,\mathrm{d}x,$$

is the mean value of u on the interval $[a,b]$.

1.6. (Inverse estimates) Let $\Omega = (a,b)$, $a < b$ be an open interval in $\mathbb{R}$.

a) Show that (1.17) does not hold on $C^1(\Omega)$, i.e. there exists no constant C_Ω such that

$$\|u'\|_{2,\Omega} \leq C_\Omega \|u\|_{2,\Omega}$$

for all $u \in C^1(\Omega)$.

b) Show (1.18), i.e. prove that (1.17) holds true for all polynomials of (maximal) order p. Can you find the constant in the above estimate explicitly?

c) Consider the finite-dimensional Fourier space

$$\mathcal{F}_p(\Omega) = \left\{ u(x) = \sum_{k=0}^{p} a_k \cos(kx) + \sum_{k=1}^{p} b_k \sin(kx) : \, a_k, b_k \in \mathbb{R} \right\}.$$

Prove that the inverse estimate (1.17) holds on $\mathcal{F}_p(\Omega)$, and find the constant explicitly.

1.7. Suppose that Γ_D is given as in Example 1.35. Show that (1.38) defines a norm on the space $\{u \in C^k(\overline{\Omega}) : u = 0 \text{ on } \Gamma_D\}$ for any $k \in \mathbb{N}$.

1.8. Specify the types (according to Definition 1.22) of the following PDE:

a) Reaction-Diffusion equation:

$$u_t - \Delta u = f(u);$$

b) Hamilton-Jacobi equation:

$$u_t + H(\nabla u, \boldsymbol{x}) = 0;$$

c) p-Laplacian equation

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0.$$

1.9. Specify the types (according to Definition 1.29) of the

a) Euler-Tricomi equation,

$$yu_{xx} + u_{yy} = 0,$$

in $\mathbb{R}^2$ in dependence on y;

b) telegraph equation

$$u_{tt} + du_t - \Delta u = 0,$$

in $\mathbb{R}^m$, where $d > 0$ is a constant;

c) the linear second-order PDE

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + f = 0,$$

in $\mathbb{R}^2$ in dependence on the constant coefficients $a, b, c, d, e, f \in \mathbb{R}$.

1.10. Use separation of variables to solve the Laplace equation

$$\begin{aligned} -\Delta u(x_1, x_2) &= 0 && \text{in } \Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\} \\ u(x_1, 0) &= \frac{1}{n}\sin(nx_1) && \text{on } \partial\Omega = \{(x_1, 0) : x_1 \in \mathbb{R}\} \\ \frac{\partial u}{\partial x_2}(x_1, 0) &= 0 && \text{on } \partial\Omega = \{(x_1, 0) : x_1 \in \mathbb{R}\}. \end{aligned}$$

Is this problem well-posed as $n \to \infty$?

1.11. Find the solution $u = u(x, t)$ of the linear wave equation

$$u_{tt} - u_{xx} = 0 \qquad \text{in } (-1, 1) \times (0, \infty), \tag{1.59}$$

with boundary conditions

$$u(-1, t) = u(1, t) = 0 \qquad \forall t > 0, \tag{1.60}$$

and initial conditions

$$u(x,0) = 1 - x^2 \qquad \text{and} \quad u_t(x,0) = 0 \qquad \forall x \in (-1,1).$$

Hint: Choose a suitable representation formula for $u(x,t)$ in terms of trigonometric functions (with respect to x) and time-dependent coefficients; make sure that these functions satisfy the boundary conditions (1.60) (cf. Example 1.50). Furthermore, find an ordinary differential equation for each of the coefficients.

2

Energy Methods

Energy methods constitute an important instrument in finding qualitative properties of PDE solutions (provided that a sufficiently smooth solution exists). In this chapter, we shall focus on

- the *Dirichlet principle* which says that the solution of certain PDEs can be interpreted as minimum of a suitable energy functional. We will illustrate this with the Poisson equation (1.5);
- *stability estimates and energy properties* for PDE solutions. Specifically, we will show how, for the heat equation and the wave equation, stability bounds of the form (1.34) and related statements about suitably defined energies can be proved;
- the *uniqueness* of PDE solutions. In fact, for many linear PDEs, energy methods can be used to prove this important property of a PDE problem.

2.1 The Dirichlet Principle

Let us consider Poisson's equation (1.5) with Dirichlet boundary conditions:

$$-\Delta u = f \qquad \text{in } \Omega \tag{2.1}$$

$$u = g \qquad \text{on } \partial\Omega, \tag{2.2}$$

where $\Omega \in \mathbb{R}^m$ is a bounded open domain, and fmg are given sufficiently smooth functions. Furthermore, define the following energy functional:

$$\mathsf{E}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 \,\mathrm{d}\boldsymbol{x} - \int_\Omega fv \,\mathrm{d}\boldsymbol{x}. \tag{2.3}$$

Theorem 2.4 (Dirichlet principle) *Suppose that $u \in \mathcal{C}^2(\overline{\Omega})$ solves* (2.1)–(2.2). *Then, u minimizes the energy functional E from* (2.3) *(under the boundary condition* (2.2)*), i.e.,*

$$\mathsf{E}(u) = \min_{\substack{v\in\mathcal{C}^2(\overline{\Omega})\\ v=g \text{ on } \partial\Omega}} \mathsf{E}(v). \tag{2.5}$$

Conversely, if there exists $u \in \mathcal{C}^2(\overline{\Omega})$ that satisfies (2.5) *and the boundary condition* (2.2)*, then u is a solution of* (2.1)–(2.2).

Proof: We first show that a solution $u \in \mathcal{C}^2(\overline{\Omega})$ of (2.1)–(2.2) satisfies (2.5). To this end, consider an arbitrary function w with

$$w \in \mathcal{C}^2(\overline{\Omega}), \qquad w = g \quad \text{on } \partial\Omega. \tag{2.6}$$

Then, recalling the definition (2.3) of the energy functional E, and noticing that $-\Delta u = f$, results in

$$\mathsf{E}(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 \,\mathrm{d}x - \int_\Omega f w \,\mathrm{d}x = \frac{1}{2}\int_\Omega |\nabla w|^2 \,\mathrm{d}x + \int_\Omega w \Delta u \,\mathrm{d}x.$$

Applying Green's formula (1.4) and using that $w = g$ on $\partial\Omega$, leads to

$$\begin{aligned} \mathsf{E}(w) &= \frac{1}{2}\int_\Omega |\nabla w|^2 \,\mathrm{d}x - \int_\Omega \nabla w \cdot \nabla u \,\mathrm{d}x + \int_{\partial\Omega} (\nabla u \cdot \boldsymbol{n}) w \,\mathrm{d}s \\ &= \frac{1}{2}\int_\Omega |\nabla w|^2 \,\mathrm{d}x - \int_\Omega \nabla w \cdot \nabla u \,\mathrm{d}x + \int_{\partial\Omega} (\nabla u \cdot \boldsymbol{n}) g \,\mathrm{d}s \end{aligned} \tag{2.7}$$

for all w satisfying (2.6). Therefore, letting $w = u$ in the above identity, yields

$$\mathsf{E}(u) = -\frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_{\partial\Omega} (\nabla u \cdot \boldsymbol{n}) g \,\mathrm{d}s,$$

and hence,

$$\int_{\partial\Omega} (\nabla u \cdot \boldsymbol{n}) g \,\mathrm{d}s = \mathsf{E}(u) + \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x.$$

Inserting this into (2.7), we obtain

$$\begin{aligned} \mathsf{E}(w) &= \mathsf{E}(u) + \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla w|^2 \,\mathrm{d}x - \int_\Omega \nabla w \cdot \nabla u \,\mathrm{d}x \\ &\geq \mathsf{E}(u) + \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla w|^2 \,\mathrm{d}x - \int_\Omega |\nabla w||\nabla u| \,\mathrm{d}x \end{aligned}$$

Then, using (1.16) in the last term with $A = |\nabla u|$, $B = |\nabla w|$, and $\delta = 1$, we have

$$\mathsf{E}(w) \geq \mathsf{E}(u) + \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla w|^2 \,\mathrm{d}x - \left(\int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla w|^2\right) \,\mathrm{d}x\right) = \mathsf{E}(u).$$

This holds for all w satisfying (2.6), and hence, (2.5) is proved.

Let us show that (2.5) implies that u solves (2.1)–(2.2). To this end, we use a technique from the calculus of variations. Note that any function w that fulfills (2.6) can be written in the form $u + \varepsilon\varphi$, for suitable $\varepsilon \in \mathbb{R}$ and

$$\varphi \in \mathcal{C}^2(\overline{\Omega}), \qquad \varphi = 0 \text{ on } \partial\Omega. \tag{2.8}$$

Therefore, if u satisfies (2.5), it follows that the function

$$\eta(\varepsilon) := \mathsf{E}(u + \varepsilon\varphi)$$

has a minimum at $\varepsilon = 0$, and hence, $\eta'(0) = 0$. This implies that

$$\begin{aligned}
0 &= \eta'(\varepsilon)\big|_{\varepsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\frac{1}{2}\int_\Omega \nabla(u+\varepsilon\varphi)\cdot\nabla(u+\varepsilon\varphi)\,\mathrm{d}\boldsymbol{x} - \int_\Omega f(u+\varepsilon\varphi)\,\mathrm{d}\boldsymbol{x}\right)\Bigg|_{\varepsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\frac{1}{2}\varepsilon^2\int_\Omega |\nabla\varphi|^2\,\mathrm{d}\boldsymbol{x} + \varepsilon\int_\Omega \left(\nabla u\cdot\nabla\varphi - f\varphi\right)\,\mathrm{d}\boldsymbol{x} + \int_\Omega\left(\frac{1}{2}|\nabla u|^2 - fu\right)\,\mathrm{d}\boldsymbol{x}\right)\Bigg|_{\varepsilon=0} \\
&= \varepsilon\int_\Omega |\nabla\varphi|^2\,\mathrm{d}\boldsymbol{x} + \int_\Omega \left(\nabla u\cdot\nabla\varphi - f\varphi\right)\,\mathrm{d}\boldsymbol{x}\Bigg|_{\varepsilon=0} \\
&= \int_\Omega \left(\nabla u\cdot\nabla\varphi - f\varphi\right)\,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Green's formula (1.4) and the fact that $\varphi = 0$ on $\partial\Omega$ imply that

$$\begin{aligned}
0 &= \int_\Omega \nabla u\cdot\nabla\varphi\,\mathrm{d}\boldsymbol{x} - \int_\Omega f\varphi\,\mathrm{d}x \\
&= \int_\Omega (\nabla u\cdot\boldsymbol{n})\varphi\,\mathrm{d}s - \int_\Omega \varphi\Delta u\,\mathrm{d}\boldsymbol{x} - \int_\Omega f\varphi\,\mathrm{d}x \\
&= \int_\Omega (-\Delta u - f)\varphi\,\mathrm{d}\boldsymbol{x}
\end{aligned} \tag{2.9}$$

for all φ satisfying (2.8). Recalling Corollary 1.11, it follows that $-\Delta u = f$. □

Remark 2.10 A similar statement like property (2.5) holds also for the Poisson equation with Neumann boundary conditions, $\nabla u\cdot\boldsymbol{n} = g$ on $\partial\Omega$.

Remark 2.11 The Dirichlet principle can be used for numerical purposes. Here, a possible idea is to restrict (2.5) to a finite-dimensional function space V (discretization space). More precisely, a numerical approximation of the exact solution of (2.1)–(2.2) is found by solving the following optimization problem: find $\widetilde{u} \in V$ such that

$$\mathsf{E}(u) = \min_{w\in V}\mathsf{E}(w).$$

This is the so-called *Ritz-method*. The *finite element method* is closely related to this scheme.

Remark 2.12 Note that the energy functional E from (2.3) remains well-defined on spaces that are less regular than $C^2(\overline{\Omega})$ (e.g., $C^1(\Omega)$). In fact, it is possible to define "*weak solutions*" of (2.1)–(2.2) which do not belong to $C^2(\overline{\Omega})$. In addition, the discretization spaces V of Ritz or finite element methods typically consist of functions which are *not* globally C^2-regular.

2.2 Stability Estimates and Energy Properties

We will now prove some stability results for PDE solutions. More precisely, for the heat equation and the wave equation, we will show how solutions (measured in suitable norms) can be bounded by the data of the corresponding problem. In addition, we will illustrate, for some special cases, that these estimates are related to certain energy properties. We have already analyzed a corresponding estimate for the solution of the Poisson equation. Indeed, referring to (1.39), there holds

$$\|\nabla u\|_{2,\Omega} \leq C_\Omega \|f\|_{2,\Omega},$$

or, equivalently,

$$\mathsf{E}(u) \leq C_\Omega^2 \|f\|_{2,\Omega}^2.$$

where we consider the energy $\mathsf{E}(u) := \|\nabla u\|_{2,\Omega}^2$ for the Poisson problem (1.36)–(1.37) (instead of the energy (2.3) defined in the context of the Dirichlet principle).

Let now $\Omega \subset \mathbb{R}^m$ be an bounded open domain, t_0 an initial time, and $T > t$. We first consider the heat equation

$$u_t - \Delta u = f \qquad \text{for } \boldsymbol{x} \in \Omega, t \in (t_0, T), \tag{2.13}$$

with (homogeneous) Dirichlet boundary conditions

$$u(\boldsymbol{x}, t) = 0 \qquad \text{for } \boldsymbol{x} \in \partial\Omega, t \in (t_0, T), \tag{2.14}$$

and initial conditions

$$u(\boldsymbol{x}, t_0) = u_0, \qquad \boldsymbol{x} \in \Omega, \tag{2.15}$$

where u_0 is a given function. In order to ensure sufficient regularity for the following analysis, we suppose that the solution u of the above problem belongs to the space $C^{2,1}(\overline{\Omega})$, where we let $\Omega_T := \Omega \times (t_0, T)$, and, for integers $k, l \geq 0$,

$$\begin{aligned} C^{k,l}(\overline{\Omega}_T) := \{v : \overline{\Omega} \times [t_0, T] : v(\cdot, t) \in C^k(\overline{\Omega}) \text{ for all } t \in [t_0, T], \text{ and} \\ v(\boldsymbol{x}, \cdot) \in C^l([t_0, T]) \text{ for all } \boldsymbol{x} \in \overline{\Omega}\}. \end{aligned} \tag{2.16}$$

Now, multiplying the PDE with the solution u and integrating over Ω, leads to

$$\int_\Omega u_t u \,\mathrm{d}\boldsymbol{x} - \int_\Omega u \Delta u \,\mathrm{d}\boldsymbol{x} = \int_\Omega f u \,\mathrm{d}\boldsymbol{x}.$$

Applying Green's formula (1.4) to the second of the above integrals and using the zero Dirichlet boundary conditions, we obtain

$$\int_\Omega u \Delta u \,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} (\nabla u \cdot \boldsymbol{n}) u \,\mathrm{d}s - \int_\Omega |\nabla u|^2 \,\mathrm{d}\boldsymbol{x} = -\int_\Omega |\nabla u|^2 \,\mathrm{d}\boldsymbol{x}. \tag{2.17}$$

Furthermore, we notice that $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}u^2 = u_t u$. Hence, it follows that

$$\frac{1}{2}\int_\Omega \frac{\mathrm{d}}{\mathrm{d}t} u^2 \,\mathrm{d}\boldsymbol{x} + \int_\Omega |\nabla u|^2 \,\mathrm{d}\boldsymbol{x} = \int_\Omega f u \,\mathrm{d}\boldsymbol{x}. \tag{2.18}$$

Moreover, by Hölder's inequality (1.8), the Poincaré-Friedrichs inequality (1.14) and (1.16), there holds

$$\begin{aligned} \int_\Omega f u \,\mathrm{d}\boldsymbol{x} &\leq \int_\Omega |f u| \,\mathrm{d}\boldsymbol{x} \leq \|f(\cdot, t)\|_{2,\Omega} \|u(\cdot, t)\|_{2,\Omega} \leq C_\Omega \|f(\cdot, t)\| \|\nabla u(\cdot, t)\|_{2,\Omega} \\ &\leq \frac{1}{2} C_\Omega^2 \|f(\cdot, t)\|_{2,\Omega}^2 + \frac{1}{2} \|\nabla u(\cdot, t)\|_{2,\Omega}^2. \end{aligned}$$

Inserting this into the previous bound, results in

$$\frac{1}{2}\int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}u^2\,\mathrm{d}\boldsymbol{x} + \int_\Omega |\nabla u|^2\,\mathrm{d}\boldsymbol{x} \leq \frac{1}{2}C_\Omega^2 \left\|f(\cdot,t)\right\|_{2,\Omega}^2 + \frac{1}{2}\left\|\nabla u(\cdot,t)\right\|_{2,\Omega}^2,$$

and thus,

$$\frac{1}{2}\int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}u^2\,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}\boldsymbol{x} \leq \frac{1}{2}C_\Omega^2 \left\|f(\cdot,t)\right\|_{2,\Omega}^2.$$

Multiplication by 2 and integrating over time from t_0 to some $t \in (t_0,T)$, yields

$$\int_\Omega \left(u^2 - u_0^2\right)\,\mathrm{d}\boldsymbol{x} + \int_{t_0}^t \int_\Omega |\nabla u|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau \leq C_\Omega^2 \int_{t_0}^t \left\|f(\cdot,\tau)\right\|_{2,\Omega}^2\,\mathrm{d}\tau.$$

Therefore, we have proved the following result:

Proposition 2.19 *For the heat equation* (2.13)-(2.15), *there holds the stability estimate*

$$\int_\Omega u(\boldsymbol{x},t)^2\,\mathrm{d}\boldsymbol{x} + \int_{t_0}^t \left\|\nabla u(\cdot,\tau)\right\|_{2,\Omega}^2\,\mathrm{d}\tau \leq \int_\Omega u_0(\boldsymbol{x})^2\,\mathrm{d}\boldsymbol{x} + C\int_{t_0}^t \left\|f(\cdot,\tau)\right\|_{2,\Omega}^2\,\mathrm{d}\tau,$$

for all $t \in (t_0,T)$, with a constant $C > 0$ only depending on Ω. Here, we suppose that the data is sufficiently smooth, so that all of the above integrals are well-defined.

Let us consider the special case where $f \equiv 0$ on Ω, i.e. the *homogeneous* heat equation. Then, Proposition 2.19 implies that

$$\int_\Omega u^2\,\mathrm{d}\boldsymbol{x} + \int_{t_0}^t \int_\Omega |\nabla u|^2\,\mathrm{d}\boldsymbol{x} \leq \int_\Omega u_0^2\,\mathrm{d}\boldsymbol{x},$$

and thus,

$$\int_\Omega u^2\,\mathrm{d}\boldsymbol{x} \leq \int_\Omega u_0^2\,\mathrm{d}\boldsymbol{x}$$

for all $t \in (t_0,T)$ (provided that the integrals are well-defined). Hence, defining the energy

$$\mathsf{E}(t) := \int_\Omega u(\boldsymbol{x},t)^2\,\mathrm{d}\boldsymbol{x}, \qquad t \in (t_0,T), \tag{2.20}$$

there holds

$$\mathsf{E}(t) \leq \mathsf{E}(t_0) \qquad \forall t \in (t_0,T).$$

There even holds the following result

Proposition 2.21 *Suppose that $f \equiv 0$ in* (2.13). *Then, for the energy E of the homogeneous heat equation defined in* (2.20), *we have*

1. *E is monotonically decreasing, i.e. $\mathsf{E}(t_2) \leq \mathsf{E}(t_1)$ for all $t_0 \leq t_1 \leq t_2 \leq T$.*

2. E *is exponentially decaying in time, i.e. there exists a constant* $\gamma > 0$ *only depending on* Ω *such that*

$$\mathsf{E}(t) \leq \mathsf{E}(t_0)e^{-\gamma(t-t_0)}$$

for all $t \in (t_0, T)$.

Proof:

1. There holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E}(t) = \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}u^2 \,\mathrm{d}\boldsymbol{x} = 2\int_\Omega u u_t \,\mathrm{d}\boldsymbol{x}.$$

Then, due to (2.13) and using Green's formula (1.4), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E}(t) = 2\int_\Omega u\Delta u \,\mathrm{d}\boldsymbol{x} = 2\int_{\partial\Omega} u(\nabla u \cdot \boldsymbol{n}) \,\mathrm{d}s - 2\int_\Omega |\nabla u|^2 \,\mathrm{d}\boldsymbol{x} = -2\int_\Omega |\nabla u|^2 \,\mathrm{d}\boldsymbol{x}. \tag{2.22}$$

Hence, $\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E}(t) \leq 0$, which implies that E is monotonically decreasing.

2. Applying the Poincaré-Friedrichs inequality (1.14) to (2.22), there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E}(t) = -2\int_\Omega |\nabla u|^2 \,\mathrm{d}\boldsymbol{x} \leq -\frac{2}{C_\Omega^2}\int_\Omega u^2 \,\mathrm{d}\boldsymbol{x} = -\gamma\mathsf{E}(t),$$

where $\gamma = \frac{2}{C_\Omega^2} > 0$ is a constant only depending on Ω. Therefore, we obtain the following ordinary differential equation for E:

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E}(t)}{\mathsf{E}(t)} \leq -\gamma.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(\mathsf{E}(t)\right) \leq -\gamma.$$

Integration from t_0 to $t \in (t_0, T)$, yields

$$\ln\left(\mathsf{E}(t)\right) - \ln\left(\mathsf{E}(t_0)\right) \leq -\gamma(t - t_0).$$

Since $\ln$ is monotonically increasing, we find

$$\mathsf{E}(t) \leq \mathsf{E}(t_0)e^{-\gamma(t-t_0)}.$$

This completes the proof. □

We shall now turn to the homogeneous wave equation, i.e., (1.26) with $f \equiv 0$. We choose (homogeneous) Dirichlet boundary conditions like in (2.14), and initial conditions

$$u(\boldsymbol{x}, t = t_0) = u_0(\boldsymbol{x}), \quad u_t(\boldsymbol{x}, t = t_0) = u_1(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, \tag{2.23}$$

where g, h are given functions. Note that, due to the fact that the wave equation is a second-order PDE with respect to time, two initial conditions are required. We suppose that the solution fulfills

$$u \in C^{2,2}(\overline{\Omega}_T);$$

cf. (2.16). Our goal is to investigate a suitable energy for this problem. Let us therefore multiply the (homogeneous) wave equation by u_t and integrate over Ω. This results in

$$\int_\Omega u_t u_{tt} \,\mathrm{d}\boldsymbol{x} - \int_\Omega u_t \Delta u \,\mathrm{d}x = 0.$$

Due to $u = 0$ on $\partial\Omega$ for all $t \in (t_0, T)$, there holds also $u_t = 0$ on $\partial\Omega \times (t_0, T)$. Thus, integration by parts in $\boldsymbol{x}$ (Green's formula (1.4)), leads to

$$\int_\Omega u_t u_{tt} \,\mathrm{d}\boldsymbol{x} + \int_\Omega \nabla u \cdot \nabla u_t \,\mathrm{d}\boldsymbol{x} = 0.$$

Noticing that $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} u_t^2 = u_{tt} u_t$ and $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} |\nabla u|^2 = \nabla u \cdot \nabla u_t$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_t^2 \,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2 \,\mathrm{d}\boldsymbol{x} = 0.$$

Integration with respect to time from t_0 to $t \in (t_0, T)$, results in

$$\frac{1}{2}\int_\Omega \left(u_t(\boldsymbol{x},t)^2 - u_1(\boldsymbol{x})\right)^2 \,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_\Omega \left(|\nabla u(\boldsymbol{x},t)|^2 - |\nabla u_0(\boldsymbol{x})|^2\right) \,\mathrm{d}\boldsymbol{x} = 0.$$

Hence, defining the energy

$$\mathsf{E}(t) := \int_\Omega \left(u_t(\boldsymbol{x},t)^2 + |\nabla u(\boldsymbol{x},t)|^2\right) \,\mathrm{d}\boldsymbol{x}, \tag{2.24}$$

we obtain the following result:

Proposition 2.25 *For the homogeneous wave equation with homogeneous Dirichlet boundary conditions, the energy defined in* (2.24) *remains constant in time, i.e.,*

$$\mathsf{E}(t) = \mathsf{E}(t_0) = \int_\Omega \left(u_1(\boldsymbol{x})^2 + |\nabla u_0(\boldsymbol{x})|^2\right) \,\mathrm{d}\boldsymbol{x}$$

for any $t \in (t_0, T)$. Here, it is again supposed that the data is sufficiently smooth, so that all integrals are well-defined.

2.3 Uniqueness of Solutions

We will now demonstrate how energy estimates can be used to easily prove uniqueness of solutions of the corresponding PDE problems.

Theorem 2.26 *The Poisson problem* (2.1)–(2.2) *has at most one solution in $C^2(\overline{\Omega})$.*

Proof: Suppose that there are two solutions $u, \widetilde{u} \in C^2(\overline{\Omega})$ to (2.1)–(2.2). We define the difference by $v = u - \widetilde{u}$. Then, due to linearity of the PDE, there holds

$$-\Delta v = -\Delta(u - \widetilde{u}) = -\Delta u + \Delta\widetilde{u} = f - f = 0 \qquad \text{in } \Omega,$$

and

$$v = u - \widetilde{u} = g - g = 0 \qquad \text{on } \partial\Omega. \tag{2.27}$$

Due to (1.39), we have $\|\nabla v\|_{2,\Omega} = 0$. Thus, $\nabla v \equiv \mathbf{0}$, and v is constant. In addition, because of (2.27), it follows that $v \equiv 0$, i.e., $u \equiv \widetilde{u}$. ☐

Similar techniques can be applied to prove the uniqueness of the heat equation and of the wave equation. Like for the Poisson equation, we shall consider possibly nonhomogeneous (i.e., nonzero) Dirichlet boundary conditions,

$$u = g \qquad \text{on } \partial\Omega \times [t_0, T]. \tag{2.28}$$

Theorem 2.29 *The heat equation* (2.13) *with (possibly inhomogeneous) Dirichlet boundary conditions* (2.28) *and initial conditions* (2.15) *has at most one solution in* $C^{2,1}(\overline{\Omega}_T)$.

Proof: Consider two solutions u and $\widetilde{u}$ of the given PDE problem. Then, by linearity (similarly as in the proof of the previous Theorem 2.26), the difference $v = u - \widetilde{u}$ satisfies the heat problem

$$\begin{aligned} v_t - \Delta v &= 0 && \text{in } \Omega_T \\ v(\boldsymbol{x}, t) &= 0 && \text{on } \partial\Omega \times (t_0, T) \\ v(\boldsymbol{x}, t_0) &= 0 && \text{in } \Omega. \end{aligned}$$

Due to Proposition 2.21, there holds

$$\mathsf{E}(t) \leq \mathsf{E}(t_0) = \int_\Omega v(\boldsymbol{x}, t_0)^2 \, \mathrm{d}\boldsymbol{x} = 0$$

for all $t \in (t_0, T)$, where E is the energy defined in (2.20). Consequently, $v \equiv 0$ on Ω_T. ☐

Theorem 2.30 *The wave equation* (1.26) *with (possibly inhomogeneous) Dirichlet boundary conditions* (2.28) *and initial conditions* (2.23) *has at most one solution in* $C^{2,2}(\overline{\Omega}_T)$.

Proof: We define again $v = u - \widetilde{u}$, where u and $\widetilde{u}$ are two solutions of the given wave problem. Then, there holds

$$\begin{aligned} v_{tt} - \Delta v &= 0 && \text{in } \Omega_T \\ v(\boldsymbol{x}, t) &= 0 && \text{on } \partial\Omega \times (t_0, T) \\ v(\boldsymbol{x}, t_0) = v_t(\boldsymbol{x}, t_0) &= 0 && \text{in } \Omega. \end{aligned}$$

Moreover, with Proposition 2.25, we have

$$\mathsf{E}(t) = \mathsf{E}(t_0) = \int_\Omega \left(v_t(\boldsymbol{x}, t_0)^2 + |\nabla v(\boldsymbol{x}, t_0)|^2 \right) \, \mathrm{d}\boldsymbol{x} = \int_\Omega |\nabla v(\boldsymbol{x}, t_0)|^2 \, \mathrm{d}\boldsymbol{x},$$

for any $t \in (t_0, T)$, where E is the energy from (2.24). We notice that, due to $v(\boldsymbol{x}, t_0) = 0$ on Ω, it follows that $\nabla v(\boldsymbol{x}, t_0) = \mathbf{0}$ on Ω. Hence, $\mathsf{E}(t) = 0$ for all $t \in (t_0, T)$. This implies that

$$v_t(\boldsymbol{x}, t) = |\nabla v(\boldsymbol{x}, t)| = 0 \qquad \forall (\boldsymbol{x}, t) \in \Omega_T.$$

Thus, v is constant in space and time. Furthermore, since $v(\boldsymbol{x}, t_0) = 0$ for all $\boldsymbol{x} \in \Omega$, it follows that $v \equiv 0$. ☐

2.4 Exercises

2.1. On a bounded domain $\Omega \subset \mathbb{R}^m$, consider the Laplace eigenvalue problem

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

We are looking for scalars $\lambda \in \mathbb{R}$ (eigenvalues) and (nonzero, smooth) functions $u : \Omega \to \mathbb{R}$ (eigenfunctions) which satisfy the above problem.

a) Show that all eigenvalues λ are strictly positive.

b) Prove that two eigenfunctions u_1, u_2, corresponding to two different eigenvalues $\lambda_1 \neq \lambda_2$, are orthogonal with respect to the integral product, i.e.,

$$\int_\Omega u_1 u_2 \, \mathrm{d}\boldsymbol{x} = 0.$$

c) In the 1-dimensional case ($m = 1$), find the smallest eigenvalue $\lambda > 0$ on the domain $\Omega = (-1, 1)$. Here, $\partial\Omega = \{\pm 1\}$ and $\Delta u \equiv u''$.

2.2. Let $\Omega \subset \mathbb{R}^m$, $m \geq 2$, be a bounded, open domain. Suppose $u \in C^2(\overline{\Omega})$ satisfies the Neumann problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ \nabla u \cdot \boldsymbol{n} &= g && \text{on } \partial\Omega, \end{aligned}$$

where f, g are sufficiently smooth. Furthermore, for $v \in C^2(\overline{\Omega})$, define the energy functional

$$\mathsf{E}(v) = \int_\Omega \left(\frac{1}{2} |\nabla v|^2 - f v \right) \mathrm{d}\boldsymbol{x} - \int_{\partial\Omega} g v \, \mathrm{d}s.$$

Prove that $\mathsf{E}(u) = \inf_{v \in C^2(\overline{\Omega})} \mathsf{E}(v)$.

2.3. Let $\Omega \subset \mathbb{R}^m$ be a bounded domain. Furthermore, consider a *strictly positive* function $a(\boldsymbol{x}) \in C^1(\overline{\Omega})$, and a vector-valued function $\boldsymbol{b}(\boldsymbol{x}) \in C^1(\overline{\Omega})^m$. Furthermore, let $u_0(\boldsymbol{x}), f(\boldsymbol{x}, t)$ be continuous. Prove that any sufficiently smooth solution $u(\boldsymbol{x}, t)$ of the initial boundary value problem

$$\begin{aligned} u_t(\boldsymbol{x}, t) - \operatorname{div}\left(a(\boldsymbol{x}) \nabla u(\boldsymbol{x}, t)\right) - \boldsymbol{b}(\boldsymbol{x}) \cdot \nabla u(\boldsymbol{x}, t) &= f(\boldsymbol{x}, t) && \boldsymbol{x} \in \Omega, t > 0 \\ u(\boldsymbol{x}, t) &= 0 && \boldsymbol{x} \in \partial\Omega, t > 0 \\ u(\boldsymbol{x}, 0) &= u_0(\boldsymbol{x}) && \boldsymbol{x} \in \Omega, \end{aligned}$$

satisfies an inequality of the form

$$\int_\Omega |u(\boldsymbol{x}, T)|^2 \, \mathrm{d}\boldsymbol{x} \leq C(T) \left(\int_\Omega |u_0(\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x} + \int_0^T \int_\Omega |f(\boldsymbol{x}, t)|^2 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \right).$$

Hint: Consider first the function $w = e^{\alpha t} u$, with a suitable constant $\alpha \in \mathbb{R}$.

2.4. Let $\Omega \subset \mathbb{R}^m$ be a bounded, open domain, with smooth boundary $\partial\Omega$. Consider smooth functions $b(\boldsymbol{x}, t)$, $g(\boldsymbol{x})$, and $h(\boldsymbol{x})$. Moreover, let $u(\boldsymbol{x}, t) \in C^2(\overline{\Omega} \times [0, T])$, $T > 0$, be a solution of the wave equation

$$\begin{aligned} u_{tt} - \Delta u &= 0 && \text{in } \Omega \times (0, T] \\ \nabla u \cdot \boldsymbol{n} + u &= b && \text{on } \partial\Omega \times (0, T] \\ u(\boldsymbol{x}, 0) = g(\boldsymbol{x}), \quad u_t(\boldsymbol{x}, 0) &= h(\boldsymbol{x}) && \text{on } \Omega. \end{aligned}$$

Use energy methods to prove that u is the only solution in $C^2(\overline{\Omega} \times [0, T])$.
Hint: Consider two solutions $u, \tilde{u}$, define $w = u - \tilde{u}$, and use the energy

$$\mathsf{E}(t) = \frac{1}{2} \int_\Omega w_t^2 \,\mathrm{d}\boldsymbol{x} + \frac{1}{2} \int_\Omega |\nabla w|^2 \,\mathrm{d}\boldsymbol{x}.$$

Show that $\mathsf{E} \leq 0$ (and hence, $\mathsf{E}(t) \equiv 0$).

2.5. Consider an open, bounded domain $\Omega \subset \mathbb{R}^m$, and a smooth function $f : \Omega \to \mathbb{R}$ (independent of the time variable t). Furthermore, let u be a (sufficiently smooth) solution of both, the heat equation and the wave equation, i.e.,

$$u_t(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) = f(\boldsymbol{x}), \qquad u_{tt}(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) = f(\boldsymbol{x}), \qquad \text{in } \Omega \times (0, \infty), \tag{1.1}$$

with

$$u(\boldsymbol{x}, t) = 0 \qquad \text{on } \partial\Omega \times [0, \infty). \tag{1.2}$$

Prove that u is independent of t, and that there exists at most one smooth function that satisfies all the equations in (1.1)–(1.2).

2.6. Let $\Omega \subset \mathbb{R}^m$ be open and bounded. Find a suitable energy $\mathsf{E}(t)$ for the (homogeneous) telegraph equation,

$$\begin{aligned} u_{tt} + d u_t - \Delta u &= 0 && \text{in } \Omega \times (t_0, T) \\ u &= 0 && \text{on } \partial\Omega \times [t_0, T] \\ u(\boldsymbol{x}, t_0) = u_0(\boldsymbol{x}), u_t(\boldsymbol{x}, t_0) &= u_1(\boldsymbol{x}) && \text{in } \Omega, \end{aligned}$$

and prove that it decays exponentially with $t \to \infty$. Here, $d > 0$ is a constant, and u_0, u_1 are given initial conditions.

3

Poisson Equation

This chapter will discuss the solution of the Poisson equation

$$-\Delta u = f, \tag{3.1}$$

on an open domain $\Omega \subseteq \mathbb{R}^m$. Furthermore, we shall study certain properties of harmonic functions, including some mean value properties and the maximum principle.

3.1 Distributions

Distributions constitute a very useful tool, for example, in the development of solution formulas for PDEs. Here, our focus shall be to consider some of the basic concepts, without discussing the detailed aspects. More precisely, we shall show how distributions can be applied to obtain PDE solutions, and will then prove that the resulting formulas are in fact true.

3.1.1 Definition of Distributions

Distributions can be considered as generalized functions. For instance, for $\boldsymbol{x}_0 \in \mathbb{R}^m$ and $\varepsilon > 0$, let us consider the function

$$\delta_{\boldsymbol{x}_0}^{(\varepsilon)}(\boldsymbol{x}) = \begin{cases} \frac{1}{V_{m,\varepsilon}} & \text{if } \boldsymbol{x} \in B_\varepsilon(\boldsymbol{x}_0) \\ 0 & \text{otherwise} \end{cases},$$

where

$$B_\varepsilon(\boldsymbol{x}_0) = \{\boldsymbol{x} \in \mathbb{R}^m : |\boldsymbol{x} - \boldsymbol{x}_0| < \varepsilon\}, \tag{3.2}$$

is the open ball in $\mathbb{R}^m$ with center $\boldsymbol{x}_0$ and radius ε, and

$$V_{m,\varepsilon} = \int_{B_\varepsilon(\boldsymbol{x}_0)} \mathrm{d}\boldsymbol{x} = \frac{\varepsilon^m \pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}+1\right)} \tag{3.3}$$

denotes its volume (Γ is the Gamma function). Then, for any sufficiently smooth functions φ, there holds

$$\int_\Omega \delta_{\boldsymbol{x}_0}^{(\varepsilon)} \varphi \, \mathrm{d}\boldsymbol{x} = \frac{1}{V_{m,\varepsilon}} \int_{B_\varepsilon(\boldsymbol{x}_0)} \varphi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \text{average of } \varphi \text{ in } B_\varepsilon(\boldsymbol{x}_0) \xrightarrow{\varepsilon \to 0} \varphi(\boldsymbol{x}_0).$$

Hence, the “limit” $\delta_{\boldsymbol{x}_0}$ of $\delta_{x_0}^{(\varepsilon)}$ satisfies

$$\int_{\mathbb{R}^m} \delta_{\boldsymbol{x}_0} \varphi \, \mathrm{d}\boldsymbol{x} = \varphi(\boldsymbol{x}_0) \tag{3.4}$$

for all sufficiently smooth functions φ. However, the limit of $\delta_{\boldsymbol{x}_0}^{(\varepsilon)}$ is not a function in the classical sense; it is the so-called *Dirac delta distribution* at $\boldsymbol{x}_0$. In fact, it can be shown that there exists no function that satisfies (3.4) for all φ. Hence, the integral representation is to be understood in a *symbolical* sense. We will explain this in more detail in Remark 3.7.

In order to provide a mathematical framework for distributions, we shall first define convergence on the space of test functions $\mathcal{D}(\Omega)$.

Definition 3.5 *Consider a sequence* $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ *and* $\varphi \in \mathcal{D}(\Omega)$. *Then, we say that* φ_n converges *to* φ *as* $n \to \infty$,

$$\varphi_n \to \varphi, \qquad n \to \infty,$$

if there exists a compact set $M \subset \Omega$ *containing the supports of all* φ_n *and of* φ *such that* φ_n *and all of its partial derivatives of any order converge* uniformly *to* φ *and its corresponding partial derivatives as* $n \to \infty$, *i.e., given any multi-index* $\boldsymbol{\alpha} \in \mathbb{N}_0^m$, *then for all* $\varepsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *(independent of* $\boldsymbol{x} \in M$*) such that*

$$|\mathsf{D}^{\boldsymbol{\alpha}}(\varphi_n - \varphi)(\boldsymbol{x})| < \varepsilon \qquad \forall \boldsymbol{x} \in M, \forall n \geq n_0.$$

We shall now proceed with a possible definition of distributions.

Definition 3.6 *A* distribution *(generalized function) is a linear functional*

$$\ell : \mathcal{D}(\Omega) \to \mathbb{R}, \qquad \varphi \mapsto \ell(\varphi),$$

with the following continuity property: if a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ *converges to* $\varphi \in \mathcal{D}(\Omega)$, *i.e.* $\varphi_n \to \varphi$, *then* $\ell(\varphi_n) \to \ell(\varphi)$. *The space of all distributions on* Ω *is denoted by* $\mathcal{D}'(\Omega)$. *We introduce an addition and a scalar multiplication on* $\mathcal{D}'(\Omega)$:

$$\begin{aligned} (\ell_1 + \ell_2)(\varphi) &= \ell_1(\varphi) + \ell_2(\varphi), && \forall \ell_1, \ell_2 \in \mathcal{D}'(\Omega), \\ (a\ell)(\varphi) &= \ell(a\varphi), && \forall \ell \in \mathcal{D}'(\Omega), a \in \mathbb{R}. \end{aligned}$$

Furthermore, we call $\ell \in \mathcal{D}'(\Omega)$ *the zero-distribution if* $\ell(\varphi) = 0$ *for all test functions* $\varphi \in \mathcal{D}(\Omega)$. *Note that, with these definitions,* $\mathcal{D}'(\Omega)$ *is a linear space.*

Remark 3.7 We note that any integrable function $f : \Omega \to \mathbb{R}$ can be identified with a distribution

$$\varphi \mapsto \int_\Omega f(\boldsymbol{x})\varphi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}. \tag{3.8}$$

Typically, for convenience, f is used to denote both the function as well as the corresponding distribution, i.e.,

$$f(\varphi) \cong (f, \varphi) = \int_\Omega f\varphi \, \mathrm{d}\boldsymbol{x}.$$

Even if a distribution ℓ cannot be represented in terms of a function as in (3.8), we often use the (symbolical) integral form:

$$\ell(\varphi) \cong (\ell, \varphi) = \int_\Omega \ell\varphi \, \mathrm{d}\boldsymbol{x}.$$

In this sense, for example, the previously discussed Dirac delta distribution $\delta_{\boldsymbol{x}_0}$,

$$\delta_{\boldsymbol{x}_0} : \mathcal{D}(\Omega) \to \mathbb{R}, \qquad \varphi \mapsto \varphi(\boldsymbol{x}_0),$$

is often written in the form

$$\int_\Omega \delta_0(\boldsymbol{x})\varphi(\boldsymbol{x}) = (\delta_0, \varphi) = \varphi(\boldsymbol{x}_0).$$

3.1.2 Convergence in the Distributional Sense

Definition 3.9 *We say that a sequence* $\{f_n\} \subset \mathcal{D}'(\Omega)$ converges *to* $f \in \mathcal{D}'(\Omega)$ *(in the distributional sense) if*

$$(f_n, \varphi) \to (f, \varphi) \qquad \forall \varphi \in \mathcal{D}'(\Omega).$$

We note that $\mathcal{D}'(\Omega)$ is “closed” in the sense that, if (f_n, φ) converges for all $\varphi \in \mathcal{D}(\Omega)$, then there exists $f \in \mathcal{D}'(\Omega)$ such that $f_n \to f$.

Example 3.10 Consider the sequence of functions

$$f_n(x) = \sin(nx).$$

We are interested in the limit $n \to \infty$ in the distributional sense on $\mathcal{D}(\mathbb{R})$. For $\varphi \in \mathcal{D}(\Omega)$, we have

$$(f_n, \varphi) = \int_{-\infty}^{\infty} \sin(nx)\varphi(x) \, \mathrm{d}x.$$

Since φ has compact support, there holds

$$(f_n, \varphi) = \frac{1}{n} \int_{-\infty}^{\infty} \cos(nx)\varphi'(x) \, \mathrm{d}x,$$

and,

$$|(f_n, \varphi)| \leq \frac{1}{n} \int_{-\infty}^{\infty} |\cos(nx)||\varphi'(x)| \, \mathrm{d}x \leq \frac{1}{n} \int_{-\infty}^{\infty} |\varphi'(x)| \, \mathrm{d}x \overset{n\to\infty}{\longrightarrow} 0.$$

This implies that f_n converges to the zero-distribution, i.e.

$$\lim_{n\to\infty} \sin(nx) = 0$$

in the distributional sense. ♢

3.1.3 Weak Derivatives

Consider a smooth function $f \in \mathcal{D}(\Omega)$. Since f has compact support, there holds, for $1 \leq i \leq m$,

$$\int_\Omega f(\boldsymbol{x}) \frac{\partial \varphi}{\partial x_i}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = - \int_\Omega \frac{\partial f}{\partial x_i}(\boldsymbol{x}) \varphi(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \qquad \forall \varphi \in \mathcal{D}(\Omega). \tag{3.11}$$

For non-smooth functions the above equality does typically not hold since the partial derivatives $\frac{\partial f}{\partial x_i}$ might not exist, for example. In this case, we can, however, use (3.11) to introduce a new definition of derivatives. They will be called “weak derivatives”.

Definition 3.12 *Let $f \in \mathcal{D}'(\Omega)$ and $1 \leq i \leq m$. Then $g \in \mathcal{D}'(\Omega)$ is called the* weak *(or* distributional*) derivative of f in the x_i-direction, denoted by $\frac{\partial f}{\partial x_i}$, if*

$$\left(f, \frac{\partial \varphi}{\partial x_i} \right) = - \left(g, \varphi \right) \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

More general, for a multi-index $\boldsymbol{\alpha} \in \mathbb{N}_0^m$, we define the weak partial derivative corresponding to $\boldsymbol{\alpha}$ by

$$\left(\mathsf{D}^{\boldsymbol{\alpha}} f, \varphi \right) = (-1)^{|\boldsymbol{\alpha}|} \left(f, \mathsf{D}^{\boldsymbol{\alpha}} \varphi \right) \qquad \forall \varphi \in \mathcal{D}(\Omega). \tag{3.13}$$

Remark 3.14 We note that a distribution $f \in D'(\Omega)$ always has a weak derivative in $D'(\Omega)$. Indeed, for $1 \leq i \leq m$, the weak (partial) derivative with respect to the x_i direction is given by

$$\frac{\partial f}{\partial x_i}(\varphi) \cong \left(\frac{\partial f}{\partial x_i}, \varphi \right) = - \left(f, \frac{\partial \varphi}{\partial x_i} \right) \cong -f \left(\frac{\partial \varphi}{\partial x_i} \right),$$

and more general, for a multi-index $\boldsymbol{\alpha} \in \mathbb{N}_0^m$,

$$\left(\mathsf{D}^{\boldsymbol{\alpha}} f \right)(\varphi) = (-1)^{|\boldsymbol{\alpha}|} f \left(\mathsf{D}^{\boldsymbol{\alpha}} \varphi \right)$$

for any $\varphi \in \mathcal{D}(\Omega)$.

Remark 3.15 Weak derivatives are generalizations of classical derivatives. Particularly, if a function has a classical derivative on a domain Ω, then that derivative is identical to the weak derivative. We note, however, that a function with a weak derivative does not necessarily have a classical derivative.

Let us consider some examples in order to illustrate the above definition.

Example 3.16 The classical derivative of $u(x) = |x|$ is only defined piecewise on the domain $\Omega = (-1, 1)$ due to non-differentiability at $x = 0$. In particular, there exists no global classical derivative on Ω. It is possible, however, to find a weak derivative on Ω: in fact, for $\varphi \in \mathcal{D}(\Omega)$, there holds

$$\begin{aligned} \int_{-1}^1 u(x) \varphi'(x) \,\mathrm{d}x &= - \int_{-1}^0 x \varphi'(x) \,\mathrm{d}x + \int_0^1 x \varphi'(x) \,\mathrm{d}x \\ &= \int_{-1}^0 \varphi(x) - \varphi(-1) - \int_0^1 \varphi(x) \,\mathrm{d}x + \varphi(1). \end{aligned}$$

Notice that $\varphi(-1) = \varphi(1) = 0$, and hence,

$$\int_{-1}^{1} u(x)\varphi'(x)\,\mathrm{d}x = \int_{-1}^{0} \varphi(x) - \int_{0}^{1} \varphi(x)\,\mathrm{d}x = \int_{-1}^{1} \operatorname{sgn}(x)\varphi(x)\,\mathrm{d}x.$$

Therefore, the function $g(x) = \operatorname{sgn}(x)$ is the (global) weak derivative of u. Let us find the weak derivative of g. We have

$$\int_{-1}^{1} g(x)\varphi'(x)\,\mathrm{d}x = -\int_{-1}^{0} \varphi'\,\mathrm{d}x + \int_{0}^{1} \varphi'\,\mathrm{d}x = -2\varphi(0) = -\int_{-1}^{1} 2\delta_0\varphi\,\mathrm{d}x.$$

Thus, $u'' = g' = 2\delta_0$ in the weak sense. ♢

Example 3.17 Consider the function

$$u(r,\theta) = \ln r = \ln|\boldsymbol{x}|,$$

written in polar coordinates (r,θ) in $\mathbb{R}^2$. We are looking for Δu in the distributional sense. The Laplacian in polar coordinates is given by

$$\Delta v = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}; \tag{3.18}$$

cf. Exercise 3.1. Hence, we see that

$$\Delta u = 0, \qquad r \neq 0, \tag{3.19}$$

By the definition of the weak derivatives (3.13), we are looking for $g = \Delta u$ such that

$$\int_{\mathbb{R}^2} g\varphi\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^2} u\Delta\varphi\,\mathrm{d}\boldsymbol{x}$$

We split the latter integral into two parts in order to isolate the singularity of u at $r = 0$:

$$\int_{\mathbb{R}^2} u\Delta\varphi\,\mathrm{d}\boldsymbol{x} = \int_{|\boldsymbol{x}|>\varepsilon} u\Delta\varphi\,\mathrm{d}\boldsymbol{x} + \int_{|\boldsymbol{x}|\leq\varepsilon} u\Delta\varphi\,\mathrm{d}\boldsymbol{x} := I_1 + I_2.$$

Twofold integration by parts in I_1 yields

$$I_1 = \int_{|\boldsymbol{x}|>\varepsilon} \Delta u\,\varphi\,\mathrm{d}\boldsymbol{x} + \int_{|\boldsymbol{x}|=\varepsilon} (\nabla\varphi\cdot\boldsymbol{n})u\,\mathrm{d}s_{\boldsymbol{x}} - \int_{|\boldsymbol{x}|=\varepsilon} (\nabla u\cdot\boldsymbol{n})\varphi\,\mathrm{d}s_{\boldsymbol{x}},$$

where $\boldsymbol{n} = -\frac{\boldsymbol{x}}{\varepsilon}$. Due to (3.19), there holds

$$\int_{|\boldsymbol{x}|>\varepsilon} \Delta u\,\varphi\,\mathrm{d}\boldsymbol{x} = 0.$$

Furthermore,

$$\left|\int_{|\boldsymbol{x}|=\varepsilon} (\nabla\varphi\cdot\boldsymbol{n})u\,\mathrm{d}s_{\boldsymbol{x}}\right| \leq \sup_{\boldsymbol{x}\in\mathbb{R}^m} |\nabla\varphi(\boldsymbol{x})| \int_{|\boldsymbol{x}|=\varepsilon} |\ln|\boldsymbol{x}||\,\mathrm{d}s_{\boldsymbol{x}} = 2\pi\varepsilon|\ln\varepsilon| \sup_{|\boldsymbol{x}|=\varepsilon} |\nabla\varphi(\boldsymbol{x})| \xrightarrow{\varepsilon\to 0} 0.$$

Moreover,

$$\nabla u \cdot \boldsymbol{n}|_{r=\varepsilon} = -\frac{\partial u}{\partial r}\bigg|_{r=\varepsilon} = -\frac{1}{r}\bigg|_{r=\varepsilon} = -\frac{1}{\varepsilon}.$$

Therefore,

$$-\int_{|\boldsymbol{x}|=\varepsilon} (\nabla u \cdot \boldsymbol{n})\varphi \, \mathrm{d}s_{\boldsymbol{x}} = 2\pi \frac{1}{2\pi\varepsilon} \int_{|\boldsymbol{x}|=\varepsilon} \varphi \, \mathrm{d}s_{\boldsymbol{x}} \xrightarrow{\varepsilon \to 0} 2\pi\varphi(\boldsymbol{0}).$$

In addition,

$$I_2 \leq \sup_{\boldsymbol{x}\in\mathbb{R}^2} |\Delta\varphi(\boldsymbol{x})| \int_{|\boldsymbol{x}|\leq\varepsilon} |\ln|\boldsymbol{x}|| \, \mathrm{d}\boldsymbol{x},$$

and

$$\int_{|\boldsymbol{x}|\leq\varepsilon} |\ln|\boldsymbol{x}|| \, \mathrm{d}\boldsymbol{x} = \int_0^\varepsilon \mathrm{d}r \int_{|\boldsymbol{x}|=r} |\ln|\boldsymbol{x}|| \, \mathrm{d}s = 2\pi \int_0^\varepsilon r|\ln r| \, \mathrm{d}r \xrightarrow{\varepsilon \to 0} 0.$$

Summing up, we obtain

$$\int_{\mathbb{R}^2} u\Delta\varphi \, \mathrm{d}\boldsymbol{x} = 2\pi\varphi(\boldsymbol{0}),$$

and thus,

$$g = \Delta u = 2\pi\delta_0,$$

in the distributional sense. ♢

3.1.4 Sobolev Spaces

We have already mentioned that all distributions have a distributional (weak) derivative; cf. Remark 3.14. In addition to the existence of weak derivatives, it is often interesting to know whether a weak derivative is bounded in a suitable norm, especially in the case where the weak derivative of a function is itself a function. Here, so-called Sobolev spaces play an important role.

For $p \geq 1$ and $k \in \mathbb{N}$, and an open domain $\Omega \subseteq \mathbb{R}^m$, we define the Sobolev spaces

$$W^{k,p}(\Omega) = \{v \in L^p(\Omega) : \mathsf{D}^{\boldsymbol{\alpha}} v \in L^p(\Omega) \text{ for all } \boldsymbol{\alpha} \in \mathbb{N}_0^m \text{ with } |\boldsymbol{\alpha}| \leq k\},$$

where the differential operators are to be understood in the weak sense. For the definition of the L^p-spaces, we refer to (1.6). The space $W^{k,p}$ can be equipped with the norm $\|\cdot\|_{W^{k,p}(\Omega)}$ given by

$$\|v\|_{W^{k,p}(\Omega)} = \left(\|v\|_{p,\Omega}^p + \sum_{i=0}^{k} \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}_0^m \\ |\boldsymbol{\alpha}|=i}} \|\mathsf{D}^{\boldsymbol{\alpha}} v\|_{p,\Omega}^p \right)^{\frac{1}{p}}.$$

In addition, we have the following semi-norm:

$$|v|_{W^{k,p}(\Omega)} = \left(\sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}_0^m \\ |\boldsymbol{\alpha}|=k}} \|\mathsf{D}^{\boldsymbol{\alpha}} v\|_{p,\Omega}^p \right)^{\frac{1}{p}}.$$

In the case $p = 2$, the above (normed) Sobolev spaces are often denoted by $H^k(\Omega)$, i.e., for $k \in \mathbb{N}$, we set

$$H^k(\Omega) = W^{k,2}(\Omega),$$

with the corresponding norm $\|\cdot\|_{H^k(\Omega)}$ and semi-norm $|\cdot|_{H^k(\Omega)}$.

There holds the following important density result which relates smooth functions to functions in Sobolev spaces.

Theorem 3.20 *Suppose that Ω is an open bounded domain with Lipschitz boundary. Then, the space $C^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p}(\Omega)}$ for all $1 \leq p < \infty$.*

We note that functions in Sobolev spaces are not always continuous. More precisely, we have:

Theorem 3.21 *Let $\Omega \subset \mathbb{R}^m$, $m \in \mathbb{N}$, be open and bounded, with Lipschitz boundary (if $m \geq 2$). Furthermore, let $k, \varkappa \in \mathbb{N}$ with $k - \varkappa > 0$, and $p \in [1, \infty)$. Then, if*

$$\begin{cases} k - \varkappa \geq m & \text{if } p = 1, \\ k - \varkappa > \frac{m}{p} & \text{if } p > 1, \end{cases}$$

there holds[1]

$$W^{k,p} \subset C^\varkappa(\overline{\Omega}).$$

Example 3.22 The above theorem shows, for example, that for $m = 1$, there holds $H^1(\Omega) \subset C^0(\Omega)$. For $m = 2$, however, this is not anymore true (indeed, the function $u(\boldsymbol{x}) = \ln \ln \frac{e}{|\boldsymbol{x}|}$ on $\Omega = B_1(\mathbf{0})$ belongs to $H^1(\Omega)$, however, is not continuous at $\boldsymbol{x} = \mathbf{0}$). ♢

3.2 Poisson Equation in $\mathbb{R}^m$

We shall now consider the Poisson problem

$$-\Delta u = f \qquad \text{in } \mathbb{R}^m. \tag{3.23}$$

In order to derive a solution formula, let us suppose that there exists a function Φ that satisfies

$$-\Delta \Phi = \delta_0, \tag{3.24}$$

where δ_0 is the delta distribution (at 0) from (3.4). Then, for some function v, there holds

$$v(\boldsymbol{x}) = \int_{\mathbb{R}^m} v(\boldsymbol{x} - \boldsymbol{y}) \delta_0(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} = -\int_{\mathbb{R}^m} v(\boldsymbol{x} - \boldsymbol{y}) \Phi(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}.$$

Then, *formally* using Green's formula and supposing that all functions involved vanish towards infinity, results in

$$v(\boldsymbol{x}) = \int_{\mathbb{R}^m} \nabla_{\boldsymbol{y}} v(\boldsymbol{x} - \boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} \Phi(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} = -\int_{\mathbb{R}^m} \Delta_{\boldsymbol{y}} v(\boldsymbol{x} - \boldsymbol{y}) \Phi(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}.$$

---

[1] More precisely, for every $u \in W^{k,p}(\Omega)$, there is a $\widetilde{u} \in C^\varkappa(\overline{\Omega})$ which coincides with u almost everywhere, i.e., up to a zero-measure set (in the sense of the Lebesgue integration).

Particularly, for the solution u of the Poisson problem (3.23), we (formally) obtain the solution formula

$$u(\boldsymbol{x}) = -\int_{\mathbb{R}^m} \Delta_{\boldsymbol{y}} u(\boldsymbol{x}-\boldsymbol{y})\Phi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}^m} f(\boldsymbol{x}-\boldsymbol{y})\Phi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}. \tag{3.25}$$

Evidently, the above representation for a solution of (3.23) still requires the following questions to be answered:

1. What is the solution of (3.24) in $\mathbb{R}^m$? Here, Example 3.17 implies that $\Phi(\boldsymbol{x}) = -\frac{1}{2\pi}\ln|\boldsymbol{x}|$ is a possible solution in $\mathbb{R}^2$.
2. The derivation of a solution formula was rather formal. Hence, once a function Φ is found, we need to check that (3.25) is in fact a solution of (3.23).

These two issues shall be addressed in the following two sections.

3.2.1 Fundamental Solution

We are interested in finding a solution Φ of (3.24). Such a function is called a *fundamental solution* of the Laplace equation. Recalling our “construction” of the delta distribution at the beginning of Section 3.1, it is reasonable to suppose that Φ satisfies

$$\Delta\Phi(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \neq \boldsymbol{0}, \tag{3.26}$$

in the classical sense. Furthermore, let us assume that Φ is radially symmetric, i.e., there exists a function $\widetilde{\Phi}$ such that

$$\Phi(\boldsymbol{x}) = \widetilde{\Phi}(|\boldsymbol{x}|).$$

In order to solve the above equation, we shall write the Laplacian Δ in terms of the radial variable

$$r = |\boldsymbol{x}| = \sqrt{x_1^2 + x_2^2 + \ldots + x_m^2}.$$

For $1 \leq i \leq m$, there holds

$$\frac{\partial\widetilde{\Phi}}{\partial x_i} = \frac{\mathrm{d}\widetilde{\Phi}}{\mathrm{d}r}\frac{\partial r}{\partial x_i},$$

and

$$\frac{\partial^2\widetilde{\Phi}}{\partial x_i^2} = \frac{\mathrm{d}^2\widetilde{\Phi}}{\mathrm{d}r^2}\left(\frac{\partial r}{\partial x_i}\right)^2 + \frac{\mathrm{d}\widetilde{\Phi}}{\mathrm{d}r}\frac{\partial^2 r}{\partial x_i^2} = \frac{x_i^2}{r^2}\frac{\mathrm{d}^2\widetilde{\Phi}}{\mathrm{d}r^2} + \frac{r - \frac{x_i^2}{r}}{r^2}\frac{\mathrm{d}\widetilde{\Phi}}{\mathrm{d}r}.$$

Therefore,

$$\Delta\widetilde{\Phi} = \sum_{i=1}^m \frac{\partial^2\widetilde{\Phi}}{\partial x_i^2} = \frac{1}{r^2}\frac{\mathrm{d}^2\widetilde{\Phi}}{\mathrm{d}r^2}\sum_{i=1}^m x_i^2 + \frac{1}{r^2}\frac{\mathrm{d}\widetilde{\Phi}}{\mathrm{d}r}\sum_{i=1}^m\left(r - \frac{x_i^2}{r}\right) = \frac{\mathrm{d}^2\widetilde{\Phi}}{\mathrm{d}r^2} + \frac{m-1}{r}\frac{\mathrm{d}\widetilde{\Phi}}{\mathrm{d}r}.$$

Thus

$$\Delta\widetilde{\Phi}(r) = \widetilde{\Phi}''(r) + \frac{m-1}{r}\widetilde{\Phi}'(r),$$

and (3.26) becomes

$$0 = \widetilde{\Phi}''(r) + \frac{m-1}{r}\widetilde{\Phi}'(r).$$

Hence, provided that $\widetilde{\Phi}' \neq 0$,

$$\frac{\widetilde{\Phi}''}{\widetilde{\Phi}'} = \frac{1-m}{r}.$$

Noticing that $\left(\ln \widetilde{\Phi}'\right)' = \frac{\widetilde{\Phi}''}{\widetilde{\Phi}'}$, leads to

$$\ln \widetilde{\Phi}' = (1-m)\ln r + C,$$

and

$$\widetilde{\Phi}' = \widetilde{C} r^{1-m},$$

where $C, \widetilde{C} \in \mathbb{R}$ are constants. Therefore,

$$\widetilde{\Phi}(r) = \begin{cases} a \ln r + b & \text{if } m = 2 \\ a\frac{1}{r^{m-2}} + b & \text{if } m \geq 3, \end{cases}$$

where $a, b \in \mathbb{R}$ are arbitrary constants. Letting $b = 0$ and choosing a such that $\Phi(\boldsymbol{x}) = \widetilde{\Phi}(|\boldsymbol{x}|)$ satisfies (3.24), leads to the following definition:

Definition 3.27 *The function*

$$\Phi(\boldsymbol{x}) = \begin{cases} -\frac{1}{2\pi} \ln |\boldsymbol{x}| & \text{if } m = 2 \\ \frac{1}{m(m-2)V(m)} \frac{1}{|\boldsymbol{x}|^{m-2}} & \text{if } m \geq 3, \end{cases} \tag{3.28}$$

is called the fundamental solution *of the Laplace equation. Here, $V_{m,1}$ is the volume of the unit ball in $\mathbb{R}^m$; cf.* (3.3).

Proposition 3.29 *The fundamental solution from* (3.28) *satisfies* (3.24) *in the distributional (weak) sense.*

Proof: See Exercise 3.2. □

3.2.2 Solution of $-\Delta u = f$ in $\mathbb{R}^m$

Theorem 3.30 *Let $f \in \mathcal{C}_c^2(\mathbb{R}^m)$ in* (3.1). *Then, the function*

$$u(\boldsymbol{x}) = \int_{\mathbb{R}^m} \Phi(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \tag{3.31}$$

is well-defined, belongs to $C^2(\mathbb{R}^m)$, and satisfies the Poisson equation (3.1). *Here, Φ is the fundamental solution* (3.28) *of the Laplace equation.*

Proof: See Evans, pp. 23–25. □

Remark 3.32 We notice that the solution of the Poisson equation in $\mathbb{R}^m$ given in (3.31) is not unique. In fact, any function of the form $v = u + \varphi$, where u is the solution from (3.31) and φ satisfies $\Delta\varphi = 0$, solves (3.23):

$$-\Delta v = -\Delta u + \Delta \varphi = f + 0 = f.$$

3.3 Poisson Equation in a Bounded Domain

We will now look at the Poisson equation in a bounded open domain Ω,

$$-\Delta u = f \qquad \text{in } \Omega. \tag{3.33}$$

In order to assure the uniqueness of a solution, we need to impose a boundary condition; cf., e.g., Chapter 2, Section 2.3. Let us consider, for example, a Dirichlet boundary condition,

$$u = g \qquad \text{on } \partial\Omega. \tag{3.34}$$

3.3.1 Green Functions

In order to obtain a representation formula for the solution of (3.33)–(3.34), we would like to proceed in a similar way as for the Poisson equation in $\mathbb{R}^m$. To this end, consider $\boldsymbol{x} \in \Omega$ and define a function $G^{\boldsymbol{x}}$ that satisfies

$$-\Delta_{\boldsymbol{y}} G^{\boldsymbol{x}}(\boldsymbol{y}) = \delta_{\boldsymbol{x}}(\boldsymbol{y}) \qquad \text{in } \Omega, \tag{3.35}$$

where $\delta_{\boldsymbol{x}}$ is the Dirac delta distribution at $\boldsymbol{x}$. Then, there holds *formally* that

$$u(\boldsymbol{x}) = \int_\Omega \delta_{\boldsymbol{x}}(\boldsymbol{y}) u(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} = -\int_\Omega \Delta_{\boldsymbol{y}} G^{\boldsymbol{x}}(\boldsymbol{y}) u(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}.$$

Furthermore, twofold formal integration by parts yields

$$\begin{aligned} u(\boldsymbol{x}) &= -\int_{\partial\Omega} (\nabla_{\boldsymbol{y}} G^{\boldsymbol{x}}(\boldsymbol{y}) \cdot \boldsymbol{n}) u(\boldsymbol{y}) \,\mathrm{d}s_{\boldsymbol{y}} + \int_\Omega \nabla_{\boldsymbol{y}} G^{\boldsymbol{x}}(\boldsymbol{y}) \cdot \nabla u(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} \\ &= -\int_{\partial\Omega} (\nabla_{\boldsymbol{y}} G^{\boldsymbol{x}}(\boldsymbol{y}) \cdot \boldsymbol{n}) u(\boldsymbol{y}) \,\mathrm{d}s_{\boldsymbol{y}} + \int_{\partial\Omega} G^{\boldsymbol{x}}(\boldsymbol{y}) (\nabla u(\boldsymbol{y}) \cdot \boldsymbol{n}) \,\mathrm{d}s_{\boldsymbol{y}} - \int_\Omega G^{\boldsymbol{x}}(\boldsymbol{y}) \Delta u(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}. \end{aligned}$$

Using (3.33)–(3.34) results in

$$u(\boldsymbol{x}) = -\int_{\partial\Omega} (\nabla_{\boldsymbol{y}} G^{\boldsymbol{x}}(\boldsymbol{y}) \cdot \boldsymbol{n}) g(\boldsymbol{y}) \,\mathrm{d}s_{\boldsymbol{y}} + \int_{\partial\Omega} G^{\boldsymbol{x}}(\boldsymbol{y}) (\nabla u(\boldsymbol{y}) \cdot \boldsymbol{n}) \,\mathrm{d}s_{\boldsymbol{y}} + \int_\Omega G^{\boldsymbol{x}}(\boldsymbol{y}) f(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}. \tag{3.36}$$

Provided that we can find the function $G^{\boldsymbol{x}}$, we see that the above expression is already almost a solution formula for (3.33)–(3.34). We remark, however, that we do not have any information on the normal derivative $\nabla u \cdot \boldsymbol{n}$ of u along $\partial\Omega$. The problem can easily be resolved by imposing the following homogeneous (Dirichlet) boundary condition on $G^{\boldsymbol{x}}$:

$$G^{\boldsymbol{x}} = 0 \qquad \text{on } \partial\Omega. \tag{3.37}$$

With this, there holds

$$u(\boldsymbol{x}) = -\int_{\partial\Omega} (\nabla_{\boldsymbol{y}} G^{\boldsymbol{x}}(\boldsymbol{y}) \cdot \boldsymbol{n}) g(\boldsymbol{y}) \,\mathrm{d}s_{\boldsymbol{y}} + \int_\Omega G^{\boldsymbol{x}}(\boldsymbol{y}) f(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}. \tag{3.38}$$

This is now indeed a solution formula.

Remark 3.39 We emphasize that the above calculation was formal, and hence, the solution formula (3.38) needs to be verified carefully.

Remark 3.40 The function $G^{\boldsymbol{x}}$ does only depend on the domain Ω and not on the data f and g. In particular, once $G^{\boldsymbol{x}}$ is known for a given domain Ω, the solution formula (3.38) for the Poisson problem on Ω can be used for any (sufficiently smooth) data f, g.

Let us now discuss the function $G^{\boldsymbol{x}}$. In order to fulfill (3.35), the function

$$\widetilde{G}^{\boldsymbol{x}}(\boldsymbol{y}) = \Phi(\boldsymbol{x} - \boldsymbol{y})$$

is a plausible choice; cf. (3.24). Evidently, however, $\widetilde{G}^{\boldsymbol{x}}$ does not satisfy the homogeneous boundary conditions (3.37). For this reason we subtract a *boundary correction function* $\varphi^{\boldsymbol{x}}$ from $\widetilde{G}^{\boldsymbol{x}}$. More precisely, we let

$$G^{\boldsymbol{x}}(\boldsymbol{y}) = \Phi(\boldsymbol{x} - \boldsymbol{y}) - \varphi^{\boldsymbol{x}}(\boldsymbol{y}). \tag{3.41}$$

Then, in order for (3.35), (3.37) to hold, we require

$$-\Delta_{\boldsymbol{y}}\varphi^{\boldsymbol{x}}(\boldsymbol{y}) = -\Delta_{\boldsymbol{y}}\Phi(\boldsymbol{x} - \boldsymbol{y}) + \Delta_{\boldsymbol{y}}G^{\boldsymbol{x}}(\boldsymbol{y}) = \delta_{\boldsymbol{x}}(\boldsymbol{y}) - \delta_{\boldsymbol{x}}(\boldsymbol{y}) = 0,$$

in $\partial\Omega$, and

$$\varphi^{\boldsymbol{x}}(\boldsymbol{y}) = \Phi(\boldsymbol{x} - \boldsymbol{y}) - G^{\boldsymbol{x}}(\boldsymbol{y}) = \Phi(\boldsymbol{x} - \boldsymbol{y})$$

on $\partial\Omega$. This leads to the following definition.

Definition 3.42 *For fixed $\boldsymbol{x} \in \Omega$, the function $G^{\boldsymbol{x}}$ from* (3.35)–(3.37) *is called* Green function *for the Poisson problem* (3.33)–(3.34)*. It is represented by*

$$G^{\boldsymbol{x}}(\boldsymbol{y}) = \Phi(\boldsymbol{x} - \boldsymbol{y}) - \varphi^{\boldsymbol{x}}(\boldsymbol{y}),$$

where Φ is the fundamental solution of the Laplace equation from (3.28)*, and $\varphi^{\boldsymbol{x}}$ is a boundary correction function satisfying*

$$-\Delta_{\boldsymbol{y}}\varphi^{\boldsymbol{x}}(\boldsymbol{y}) = 0 \qquad \boldsymbol{y} \in \Omega \tag{3.43}$$

$$\varphi^{\boldsymbol{x}}(\boldsymbol{y}) = \Phi(\boldsymbol{x} - \boldsymbol{y}) \qquad \boldsymbol{y} \in \partial\Omega. \tag{3.44}$$

Here, we suppose that $\varphi^{\boldsymbol{x}} \in C^2(\overline{\Omega})$.

Remark 3.45

1. We note that the function $\Phi(\boldsymbol{x} - \boldsymbol{y})$ prescribing the values of the boundary correction function on $\partial\Omega$ is well-defined. Indeed, for $\boldsymbol{x} \in \Omega$ and $\boldsymbol{y} \in \partial\Omega$, we clearly have $\boldsymbol{x} \neq \boldsymbol{y}$, and thus, the singularity of Φ at $\mathbf{0}$ is not a source of difficulty.

2. The Green function is symmetric, i.e.,

   $$G^{\boldsymbol{x}}(\boldsymbol{y}) = G^{\boldsymbol{y}}(\boldsymbol{x}) \qquad \forall \boldsymbol{x}, \boldsymbol{y} \in \Omega;$$

   cf. Exercise 3.4.

3. A similar solution formula like in (3.38) can be obtained for the Poisson problems with other boundary conditions than the Dirichlet boundary condition (3.34) (e.g., Neumann or mixed boundary conditions). To this end, the boundary conditions for the Green function $G^{\boldsymbol{x}}$ have to be chosen appropriately so that (the parts of) the integrals in (3.36) for which no boundary data is given vanish.

4. The idea of representing the solution of the Poisson problem in terms of a Green function extends to more general PDE problems in a natural way.

3.3.2 Solution of $-\Delta u = f$ in a Bounded Domain

Using similar techniques as in the proof of Theorem 3.30 for the Poisson problem in $\mathbb{R}^m$, it can been shown that the solution u from (3.38) satisfies (3.33). In addition, it needs to proved that u fulfills the boundary condition g at the boundary $\partial\Omega$. More precisely, there holds the following result.

Theorem 3.46 *Let $\Omega \subset \mathbb{R}^m$ be open and bounded. Furthermore, suppose that* (3.43)–(3.44) *and* (3.33)–(3.34) *have solutions $\varphi^{\boldsymbol{x}}$ (for any $\boldsymbol{x} \in \Omega$) and u, respectively, which both belong to $C^2(\overline{\Omega})$. Then, u is given by* (3.38). *Here, u from* (3.38) *satisfies the Dirichlet boundary condition* (3.34) *in the following limit sense: Given a convergent sequence $\{\boldsymbol{x}_n\}_{n\geq\mathbb{N}} \subset \Omega$ with a limit $\overline{\boldsymbol{x}} \in \partial\Omega$, then $\lim_{n\to\infty} u(\boldsymbol{x}_n) = g(\overline{\boldsymbol{x}})$.*

3.3.3 Examples of Green Functions

For general domains and problems, it is typically not possible to find an explicit expression (for example, in terms of elementary functions) for the associated Green function. In some special cases, however, a formula can be derived.

We shall consider the Poisson problem (with Dirichlet data) in the upper half space

$$\mathbb{H}^m_+ = \{\boldsymbol{x} \in \mathbb{R}^m : x_m > 0\},$$

and in the sphere

$$B_r(\boldsymbol{x}_0) = \{\boldsymbol{x} \in \mathbb{R}^m : |\boldsymbol{x}| < r\},$$

with radius $r > 0$ and center $\boldsymbol{x}_0 \in \mathbb{R}^m$. More examples will be discussed in Exercise 3.5.

Green Function on the Half Space

Our task is to find, for any $\boldsymbol{x} \in \mathbb{H}^m_+$, a function $\varphi^{\boldsymbol{x}}$ that satisfies (3.43)–(3.44). Then, the Green function $G^{\boldsymbol{x}}$ is given by (3.41). A tempting candidate for $\varphi^{\boldsymbol{x}}$ is the function $\Phi(\boldsymbol{x} - \boldsymbol{y})$. We recall, however, that this would result in a loss of the required C^2 regularity of $\varphi^{\boldsymbol{x}}$ due to the presence of a singularity at $\boldsymbol{y} = \boldsymbol{x}$. A simple remedy for this problem is to move this singularity outside of the domain $\mathbb{H}^m_+$. Clearly, this needs to be done in such a way that (3.43) and (3.44) are still satisfied. The reflection

$$\mathsf{R} : \boldsymbol{x} = (x_1, x_2, \ldots, x_{m-1}, x_m) \mapsto \mathsf{R}(\boldsymbol{x}) = (x_1, x_2, \ldots, x_{m-1}, -x_m),$$

does the trick. Indeed, by setting

$$\varphi^{\boldsymbol{x}}(\boldsymbol{y}) = \Phi(\mathbf{R}(\boldsymbol{x}) - \boldsymbol{y}),$$

we see that $\varphi^{\boldsymbol{x}} \in C^2(\mathbb{H}_+^m)$ since $\mathbf{R}(\boldsymbol{x}) \notin \mathbb{H}_+^m$ for $\boldsymbol{x} \in \mathbb{H}_+^m$, and hence, $\mathbf{R}(\boldsymbol{x}) \neq \boldsymbol{y}$ for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{H}_+^m$. In particular, there holds

$$-\Delta_{\boldsymbol{y}} \Phi(\mathbf{R}(\boldsymbol{x}) - \boldsymbol{y}) = 0$$

in the classical sense. In addition, for $\boldsymbol{x} \in \mathbb{H}_+^m$ and $\boldsymbol{y} \in \partial\mathbb{H}_+^m$, where

$$\partial\mathbb{H}_+^m = \{\boldsymbol{x} \in \mathbb{R}^m : \, x_m = 0\},$$

we have that $|\mathbf{R}(\boldsymbol{x}) - \boldsymbol{y}| = |\boldsymbol{x} - \boldsymbol{y}|$, and thus,

$$\varphi^{\boldsymbol{x}}(\boldsymbol{y}) = \Phi(\mathbf{R}(\boldsymbol{x}) - \boldsymbol{y}) = \Phi(\boldsymbol{x} - \boldsymbol{y}).$$

Therefore, (3.43)–(3.44) are satisfied.

Let us consider the Laplace equation on $\mathbb{H}_+^m$ with Dirichlet boundary conditions:

$$-\Delta u = 0 \qquad \text{in } \mathbb{H}_+^m \tag{3.47}$$

$$u = g \qquad \text{on } \partial\mathbb{H}_+^m, \tag{3.48}$$

for some given boundary data g. Then, using (3.38), the solution is given by

$$u(\boldsymbol{x}) = -\int_{\partial\mathbb{H}_+^m} (\nabla_{\boldsymbol{y}} G^{\boldsymbol{x}}(\boldsymbol{y}) \cdot \boldsymbol{n}) g(\boldsymbol{y}) \, \mathrm{d}s_{\boldsymbol{y}}. \tag{3.49}$$

On $\partial\mathbb{H}_+^m$, the normal outward vector is

$$\boldsymbol{n} = [0, \ldots, 0, -1]^\top \in \mathbb{R}^m.$$

Therefore, it follows that

$$u(\boldsymbol{x}) = \int_{\partial\mathbb{H}_+^m} \frac{\partial G^{\boldsymbol{x}}}{\partial y_m}(\boldsymbol{y}) g(\boldsymbol{y}) \, \mathrm{d}s_{\boldsymbol{y}}.$$

Recalling (3.41), there holds

$$\frac{\partial G^{\boldsymbol{x}}}{\partial y_m}(\boldsymbol{y}) = \frac{\partial}{\partial y_m} \left(\Phi(\boldsymbol{x} - \boldsymbol{y}) - \Phi(\mathbf{R}(\boldsymbol{x}) - \boldsymbol{y})\right) = -\frac{1}{mV_{m,1}} \left(\frac{y_m - x_m}{|\boldsymbol{y} - \boldsymbol{x}|} - \frac{y_m + x_m}{|\boldsymbol{y} - \mathbf{R}(\boldsymbol{x})|} \right).$$

Then, because $\boldsymbol{y} \in \partial\mathbb{H}_+^m$, we have that

$$\frac{\partial G^{\boldsymbol{x}}}{\partial y_m}(\boldsymbol{y}) = -\frac{1}{mV_{m,1}} \left(\frac{y_m - x_m}{|\boldsymbol{y} - \boldsymbol{x}|} - \frac{y_m + x_m}{|\boldsymbol{y} - \boldsymbol{x}|} \right) = \frac{1}{mV_{m,1}} \frac{2x_m}{|\boldsymbol{y} - \boldsymbol{x}|}.$$

Inserting this into (3.49), we obtain the *Poisson formula for the half space*:

$$u(\boldsymbol{x}) = \frac{2x_m}{mV_{m,1}} \int_{\partial\mathbb{H}_+^m} \frac{g(\boldsymbol{y})}{|\boldsymbol{y} - \boldsymbol{x}|^m} \, \mathrm{d}s_{\boldsymbol{y}}, \tag{3.50}$$

for the solution of (3.47)–(3.48).

Remark 3.51

1. For boundary data $g \in C^0(\partial\mathbb{H}^m_+)$ and bounded, we see that the solution of (3.47)–(3.48) is in $C^\infty(\mathbb{H}^m_+)$.
2. Notice that $\mathbb{H}^m_+$ is unbounded, and thus, Theorem 3.46 needs to be checked carefully for the half space; cf. Exercise 3.7.

Green Function for the Sphere

In order to find the Green function for the Poisson problem on $B_r(\boldsymbol{x}_0)$ with Dirichlet boundary conditions, we proceed similarly as in the previous example. Let us first consider the unit sphere $B_1(\mathbf{0})$. The boundary corrector function $\varphi^{\boldsymbol{x}}$ is, as before, expressed in terms of the fundamental solution Φ. The singularity is again removed by means of a suitable transformation. More precisely, we define the following *inversion* about the surface of the the unit ball:

$$\mathsf{I} : \overline{B_1(\mathbf{0})} \setminus \mathbf{0} \to \mathbb{R}^m \setminus B_1(\mathbf{0}), \qquad \mathsf{I}(\boldsymbol{x}) = \frac{1}{|\boldsymbol{x}|^2}\boldsymbol{x}.$$

Then, we let

$$\varphi^{\boldsymbol{x}}(\boldsymbol{y}) = \Phi((\boldsymbol{y} - \mathsf{I}(\boldsymbol{x}))|\boldsymbol{x}|).$$

Differentiation shows that the above function satisfies (3.43). Furthermore, for $\boldsymbol{y} \in \partial B_1(\mathbf{0})$, i.e., $|\boldsymbol{y}| = 1$, there holds

$$\begin{aligned}\big|(\boldsymbol{y} - \mathsf{I}(\boldsymbol{x}))|\boldsymbol{x}|\big| &= |\boldsymbol{x}|\left|\boldsymbol{y} - \frac{1}{|\boldsymbol{x}|^2}\boldsymbol{x}\right| = |\boldsymbol{x}|\sqrt{|\boldsymbol{y}|^2 - \frac{2\boldsymbol{x}\cdot\boldsymbol{y}}{|\boldsymbol{x}|^2} + \frac{1}{|\boldsymbol{x}|^2}} = |\boldsymbol{x}|\sqrt{1 - \frac{2\boldsymbol{x}\cdot\boldsymbol{y}}{|\boldsymbol{x}|^2} + \frac{1}{|\boldsymbol{x}|^2}} \\ &= \sqrt{|\boldsymbol{x}|^2 - 2\boldsymbol{x}\cdot\boldsymbol{y} + 1} = \sqrt{|\boldsymbol{x}|^2 - 2\boldsymbol{x}\cdot\boldsymbol{y} + |\boldsymbol{y}|^2} = \sqrt{(\boldsymbol{x} - \boldsymbol{y})\cdot(\boldsymbol{x} - \boldsymbol{y})} \\ &= |\boldsymbol{x} - \boldsymbol{y}|.\end{aligned}$$

Therefore,

$$\varphi^{\boldsymbol{x}}(\boldsymbol{y}) = \Phi(\boldsymbol{x} - \boldsymbol{y}) \qquad \forall \boldsymbol{y} \in \partial B_1(\mathbf{0}).$$

Consider the homogeneous problem

$$\begin{aligned} -\Delta u &= 0 \qquad \text{in } B_1(\mathbf{0}) && (3.52)\\ u &= g \qquad \text{on } \partial B_1(\mathbf{0}). && (3.53)\end{aligned}$$

Then, applying (3.38), leads to the solution formula

$$u(\boldsymbol{x}) = -\int_{\partial B_1(\mathbf{0})} (\nabla_{\boldsymbol{y}} G^{\boldsymbol{x}}(\boldsymbol{y}) \cdot \boldsymbol{n}) g(\boldsymbol{y})\,\mathrm{d}s_{\boldsymbol{y}},$$

with

$$G^{\boldsymbol{x}}(\boldsymbol{y}) = \Phi(\boldsymbol{x} - \boldsymbol{y}) - \Phi((\boldsymbol{y} - \mathsf{I}(\boldsymbol{x}))|\boldsymbol{x}|).$$

Noticing that $\boldsymbol{n} = \boldsymbol{y}$ on $\partial B_1(\mathbf{0})$, we obtain

$$u(\boldsymbol{x}) = -\int_{\partial B_1(\mathbf{0})} (\nabla_{\boldsymbol{y}} G^{\boldsymbol{x}}(\boldsymbol{y}) \cdot \boldsymbol{y}) g(\boldsymbol{y}) \, \mathrm{d}s_{\boldsymbol{y}}.$$

A few calculations show that

$$\nabla_{\boldsymbol{y}} G^{\boldsymbol{x}}(\boldsymbol{y}) \cdot \boldsymbol{y} = -\frac{1}{V_{m,1}} \frac{1 - |\boldsymbol{x}|^2}{|\boldsymbol{x} - \boldsymbol{y}|^m}$$

for $\boldsymbol{y} \in \partial B_1(\mathbf{0})$. Hence,

$$u(\boldsymbol{x}) = \frac{1 - |\boldsymbol{x}|^2}{m V_{m,1}} \int_{\partial B_1(\mathbf{0})} \frac{g(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^m} \, \mathrm{d}s_{\boldsymbol{y}}, \tag{3.54}$$

which is the *Poisson formula* for the unit sphere.

In order to find a solution formula for (3.52)–(3.53) on the general sphere $B_r(\boldsymbol{x}_0)$, we apply a *scaling argument*. To this end, define an affine mapping

$$\boldsymbol{F} : \, B_1(\mathbf{0}) \to B_r(\boldsymbol{x}_0), \qquad \widehat{\boldsymbol{x}} \mapsto \boldsymbol{x} = r\widehat{\boldsymbol{x}} + \boldsymbol{x}_0.$$

Furthermore, let

$$\widehat{u}(\widehat{\boldsymbol{x}}) = u(\boldsymbol{F}(\widehat{\boldsymbol{x}})) = u(r\widehat{\boldsymbol{x}} + \boldsymbol{x}_0).$$

Then, there holds

$$\begin{aligned} -\Delta_{\widehat{\boldsymbol{x}}} \widehat{u} &= 0 && \text{in } B_1(\mathbf{0}) \\ \widehat{u}(\widehat{\boldsymbol{x}}) &= g(\boldsymbol{F}(\widehat{\boldsymbol{x}})) && \text{on } \partial B_1(\mathbf{0}), \end{aligned}$$

i.e., formula (3.54) is applicable. There holds

$$\begin{aligned} u(\boldsymbol{x}) = u(\boldsymbol{F}(\widehat{\boldsymbol{x}})) = \widehat{u}(\widehat{\boldsymbol{x}}) &= \frac{1 - |\widehat{\boldsymbol{x}}|^2}{m V_{m,1}} \int_{\partial B_1(\mathbf{0})} \frac{g(\boldsymbol{F}(\widehat{\boldsymbol{y}}))}{|\widehat{\boldsymbol{x}} - \widehat{\boldsymbol{y}}|^m} \, \mathrm{d}s_{\widehat{\boldsymbol{y}}} \\ &= \frac{1 - \left|\frac{\boldsymbol{x} - \boldsymbol{x}_0}{r}\right|^2}{m V_{m,1}} \int_{\partial B_1(\mathbf{0})} \frac{g(\boldsymbol{F}(\widehat{\boldsymbol{y}}))}{\left|\frac{\boldsymbol{x} - \boldsymbol{x}_0}{r} - \widehat{\boldsymbol{y}}\right|^m} \, \mathrm{d}s_{\widehat{\boldsymbol{y}}}. \end{aligned}$$

Using the variable transform

$$\boldsymbol{y} = \boldsymbol{F}(\widehat{\boldsymbol{y}}), \qquad \mathrm{d}s_{\boldsymbol{y}} = r^{m-1} \, \mathrm{d}s_{\widehat{\boldsymbol{y}}},$$

leads to the solution

$$u(\boldsymbol{x}) = \frac{r^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2}{r m V_{m,1}} \int_{\partial B_r(\boldsymbol{x}_0)} \frac{g(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^m} \, \mathrm{d}s_{\boldsymbol{y}}$$

of (3.33)–(3.34) for $\Omega = B_r(\boldsymbol{x}_0)$.

3.4 Harmonic Functions

In this section we will study functions $u \in C^2(\Omega)$, where $\Omega \subseteq \mathbb{R}^m$ is an open domain, that satisfy the Laplace equation

$$-\Delta u = 0 \qquad \text{on } \Omega.$$

These functions are called *harmonic* on Ω. We shall discuss some of their (many) interesting properties in the following.

3.4.1 Mean Value Properties

Given a harmonic function u on an open domain $\Omega \subseteq \mathbb{R}^m$, let us consider a ball $B_r(\boldsymbol{x}) \subset \Omega$ (such that $\overline{B_r(\boldsymbol{x})} \subset \Omega$) with radius $r > 0$ and center $\boldsymbol{x} \in \Omega$. Then, using (3.54) (with $\boldsymbol{x} = \boldsymbol{x}_0$), there holds

$$u(\boldsymbol{x}) = \frac{r}{mV_{m,1}} \int_{\partial B_r(\boldsymbol{x})} \frac{u(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^m} \, \mathrm{d}s_{\boldsymbol{y}}.$$

Furthermore, noticing that $|\boldsymbol{x}-\boldsymbol{y}| = r$ for all $\boldsymbol{y} \in \partial B_r(\boldsymbol{x})$, leads to

$$u(\boldsymbol{x}) = \frac{1}{mV_{m,1}r^{m-1}} \int_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{y}) \, \mathrm{d}s_{\boldsymbol{y}} = ⨍_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{y}) \, \mathrm{d}s_{\boldsymbol{y}}.$$

Here, we use the symbol $⨍$ to denote the average integral over the corresponding domain. The above equality shows that the value of a harmonic function at some point $\boldsymbol{x}$ is the average of the function over the surface of a ball $B_r(\boldsymbol{x})$ with arbitrary radius $r > 0$ (provided that $\overline{B_r(\boldsymbol{x})} \subset \Omega$). More precisely there holds:

Proposition 3.55 *Let Ω be open and $u \in C^2(\Omega)$ harmonic. Then, for all $\boldsymbol{x} \in \Omega$ and $r > 0$ such that $\overline{B_r(\boldsymbol{x})} \subset \Omega$, there holds*

$$u(\boldsymbol{x}) = ⨍_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{y}) \, \mathrm{d}s_{\boldsymbol{y}}. \tag{3.56}$$

On the other hand, if $u \in C^2(\Omega)$ satisfies the above equality (3.56) *for all balls $\overline{B_r(\boldsymbol{x})} \subset \Omega$, then u is harmonic in Ω.*

Proof: It remains to prove that (3.56) implies that u is harmonic. This shall be done by contradiction. Suppose that $\Delta u(\boldsymbol{x}_0) \neq 0$ at some point $\boldsymbol{x}_0$ in Ω. Then, because $u \in C^2(\Omega)$ (and hence $\Delta u \in C^0(\Omega)$), there exists $\varepsilon > 0$ such that $\Delta u > 0$ or $\Delta u < 0$ on $\overline{B_\varepsilon(\boldsymbol{x}_0)} \subset \Omega$. Without loss of generality, let us suppose that $\Delta u > 0$ on $B_\varepsilon(\boldsymbol{x}_0)$. Then, by assumption, there holds

$$u(\boldsymbol{x}_0) = ⨍_{\partial B_r(\boldsymbol{x}_0)} u(\boldsymbol{y}) \, \mathrm{d}s_{\boldsymbol{y}}$$

for all $r \in (0, \varepsilon)$. Due to the fact that the left-hand side of the above equality is independent of r, we have

$$0 = \frac{\partial}{\partial r} ⨍_{\partial B_r(\boldsymbol{x}_0)} u(\boldsymbol{y}) \, \mathrm{d}s_{\boldsymbol{y}} = \frac{\partial}{\partial r} \left(\frac{1}{mV_{m,1}r^{m-1}} \int_{\partial B_r(\boldsymbol{x}_0)} u(\boldsymbol{y}) \, \mathrm{d}s_{\boldsymbol{y}} \right).$$

In order to calculate the above derivative, it is convenient to use the following change of variables to the surface of the unit sphere $\partial B_1(\mathbf{0}) \subset \mathbb{R}^m$:

$$\boldsymbol{y} = r\widehat{\boldsymbol{y}} + \boldsymbol{x}, \qquad \mathrm{d}s_{\boldsymbol{y}} = r^{m-1}\,\mathrm{d}s_{\widehat{\boldsymbol{y}}}.$$

This yields

$$0 = \frac{\partial}{\partial r}\left(\frac{1}{mV_{m,1}}\int_{\partial B_1(\mathbf{0})} u(r\widehat{\boldsymbol{y}} + \boldsymbol{x})\,\mathrm{d}s_{\widehat{\boldsymbol{y}}}\right) = \frac{1}{mV_{m,1}}\int_{\partial B_1(\mathbf{0})} \nabla_{\widehat{\boldsymbol{y}}} u(r\widehat{\boldsymbol{y}} + \boldsymbol{x}) \cdot \widehat{\boldsymbol{y}}\,\mathrm{d}s_{\widehat{\boldsymbol{y}}}.$$

Transforming back to $\partial B_r(\boldsymbol{x}_0)$, result in

$$0 = \frac{1}{mV_{m,1}r^{m-1}}\int_{\partial B_r(\boldsymbol{x}_0)} \nabla_{\boldsymbol{y}} u(\boldsymbol{y}) \cdot \frac{\boldsymbol{y} - \boldsymbol{x}_0}{r}\,\mathrm{d}s_{\boldsymbol{y}}.$$

Noticing that $\frac{\boldsymbol{y}-\boldsymbol{x}_0}{r} = \boldsymbol{n}$ on $\partial B_r(\boldsymbol{x}_0)$ and applying Green's formula (1.4), we obtain

$$0 = \frac{1}{mV_{m,1}r^{m-1}}\int_{\partial B_r(\boldsymbol{x}_0)} \nabla_{\boldsymbol{y}} u(\boldsymbol{y}) \cdot \boldsymbol{n}\,\mathrm{d}s_{\boldsymbol{y}} = \frac{1}{mV_{m,1}r^{m-1}}\int_{\partial B_r(\boldsymbol{x}_0)} \Delta u(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.$$

This, however, is a contradiction to the fact that $\Delta u > 0$ on $\partial B_r(\boldsymbol{x}_0)$. ☐

Remark 3.57 The equality (3.56) involves an average integral over the surface of some ball $\partial B_r(\boldsymbol{x})$. Incidentally, this integral can be replaced with an average integral over the whole ball $B_r(\boldsymbol{x})$, i.e.,

$$u(\boldsymbol{x}) = \fint_{B_r(\boldsymbol{x})} u(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y};$$

see Exercise 3.9.

Let us investigate the gradient of a harmonic function $u \in C^2(\Omega)$ on Ω. To this end, we compute, for $i = 1, 2, \ldots, m$, the first-order partial derivative of u in the i^{th} coordinate direction. Using Proposition 3.55, there holds

$$\frac{\partial u}{\partial x_i}(\boldsymbol{x}_0) = \fint_{\partial B_r(\boldsymbol{x}_0)} u_{x_i}(\boldsymbol{y})\,\mathrm{d}s_{\boldsymbol{y}},$$

where $\boldsymbol{x}_0 \in \Omega$ and $r > 0$ such that $\overline{B_r(\boldsymbol{x}_0)} \subset \Omega$. Then, defining the vector function $\boldsymbol{v} = (v_1, v_2, \ldots, v_m)$ by

$$v_j(\boldsymbol{y}) = \begin{cases} u(\boldsymbol{y}) & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}, \qquad \boldsymbol{y} \in \overline{B_r(\boldsymbol{x}_0)},$$

there holds $u_{x_i} = \mathrm{div}\boldsymbol{v}$. Hence, using the Green formula (1.4), yields

$$\begin{aligned}\frac{\partial u}{\partial x_i}(\boldsymbol{x}_0) &= \frac{1}{V_{m,r}}\int_{B_r(\boldsymbol{x}_0)} \mathrm{div}\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \frac{1}{V_{m,r}}\int_{\partial B_r(\boldsymbol{x}_0)} \boldsymbol{v}(\boldsymbol{y}) \cdot \boldsymbol{n}\,\mathrm{d}s_{\boldsymbol{y}} \\ &= \frac{1}{V_{m,r}}\int_{\partial B_r(\boldsymbol{x}_0)} u(\boldsymbol{y}) n_i\,\mathrm{d}s_{\boldsymbol{y}},\end{aligned}$$

where $\boldsymbol{n} = (n_1, n_2, \ldots, n_m)^\top \in \mathbb{R}^m$ is the unit outward vector on $\partial B_r(\boldsymbol{x}_0)$. Moreover,

$$\frac{\partial u}{\partial x_i}(\boldsymbol{x}_0) = \frac{1}{V_{m,1} r^m} \int_{\partial B_r(\boldsymbol{x}_0)} u(\boldsymbol{y}) n_i \, \mathrm{d}s_{\boldsymbol{y}} = \frac{m}{r} \fint_{\partial B_r(\boldsymbol{x}_0)} u(\boldsymbol{y}) n_i \, \mathrm{d}s_{\boldsymbol{y}},$$

and thus,

$$\nabla u(\boldsymbol{x}_0) = \frac{m}{r} \fint_{\partial B_r(\boldsymbol{x}_0)} u(\boldsymbol{y}) \boldsymbol{n} \, \mathrm{d}s_{\boldsymbol{y}}. \tag{3.58}$$

3.4.2 Regularity Estimates

Let $\Omega \subset \mathbb{R}^m$ be open and bounded, and $u \in C^2(\overline{\Omega})$ a harmonic function on Ω. Now, for arbitrary $\boldsymbol{x}_0 \in \Omega$, choose $r > 0$ such that $\overline{B_r(\boldsymbol{x}_0)}$ is the largest (closed) sphere with center $\boldsymbol{x}_0$ that is contained in $\overline{\Omega}$, i.e.,

$$r = \sup \left\{ \rho > 0 : \overline{B_\rho(\boldsymbol{x}_0)} \subset \overline{\Omega} \right\}.$$

Then, the equality from (3.58) implies that

$$|\nabla u(\boldsymbol{x}_0)| \leq \frac{m}{\operatorname{dist}(\boldsymbol{x}_0, \partial\Omega)} \fint_{\partial B_r(\boldsymbol{x}_0)} |u(\boldsymbol{y})| \, \mathrm{d}s_{\boldsymbol{y}},$$

where

$$\operatorname{dist}(\boldsymbol{x}_0, \partial\Omega) = \inf_{\boldsymbol{y} \in \partial\Omega} |\boldsymbol{x}_0 - \boldsymbol{y}|$$

is the shortest distance from $\boldsymbol{x}_0$ to the boundary of Ω. Therefore,

$$|\nabla u(\boldsymbol{x}_0)| \leq \frac{m}{\operatorname{dist}(\boldsymbol{x}_0, \partial\Omega)} \sup_{\boldsymbol{x} \in \Omega} |u(\boldsymbol{x})|.$$

This is a regularity estimate. It provides a certain control for the gradient of a harmonic function. More generally, there holds:

Theorem 3.59 *Let Ω be bounded and open, and $u \in C^2(\overline{\Omega})$ harmonic on Ω. Then, for any multi-index $\boldsymbol{\alpha} \in \mathbb{N}^m$ and $\boldsymbol{x}_0$, we have*

$$|\mathsf{D}^{\boldsymbol{\alpha}}(\boldsymbol{x}_0)| \leq \left(\frac{m|\boldsymbol{\alpha}|}{\operatorname{dist}(\boldsymbol{x}_0, \partial\Omega)} \right)^{|\boldsymbol{\alpha}|} \sup_{\boldsymbol{x} \in \Omega} |u(\boldsymbol{y})|.$$

Furthermore, there holds the *Liouville Theorem*:

Theorem 3.60 *Let $u : \mathbb{R}^m \to \mathbb{R}$ be harmonic and* bounded. *Then, u is constant.*

Proof: Let $\boldsymbol{x}_0 \in \mathbb{R}^m$ and $r > 0$. Then, with (3.58), there holds

$$|\nabla u(\boldsymbol{x}_0)| \leq \frac{m}{r} \fint_{\partial B_r(\boldsymbol{x}_0)} |u(\boldsymbol{y})| \, \mathrm{d}s_{\boldsymbol{y}} \leq \frac{m}{r} \sup_{\boldsymbol{y} \in \partial B_r(\boldsymbol{x}_0)} |u(\boldsymbol{y})| \xrightarrow{r \to \infty} 0.$$

Hence, $\nabla u(\boldsymbol{x}_0) = \mathbf{0}$ for any $\boldsymbol{x}_0 \in \mathbb{R}^m$. Consequently, u is constant. □

3.4.3 Maximum Principle

A further very important property is the fact that harmonic functions on a bounded domain Ω take their maximum and minimum values on the boundary $\partial\Omega$. This is called the *maximum principle*.

Theorem 3.61 *Let $\Omega \subset \mathbb{R}^m$ be open and bounded. Consider a harmonic function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.*

- *a) Then, $\max_{\boldsymbol{x}\in\overline{\Omega}} u(\boldsymbol{x}) = \max_{\boldsymbol{x}\in\partial\Omega} u(\boldsymbol{x})$.*
- *b) If Ω is connected and there exists $\boldsymbol{x}_0 \in \Omega$ with $u(\boldsymbol{x}_0) = \max_{\boldsymbol{x}\in\overline{\Omega}} u(\boldsymbol{x})$, then u is constant in Ω.*

Proof: Note that b) implies a), and hence, we shall only show b). Suppose that there exists $\boldsymbol{x}_0 \in \Omega$ such that $u(\boldsymbol{x}_0) = \max_{\boldsymbol{x}\in\overline{\Omega}} u(\boldsymbol{x})$. Denote this maximum by $\overline{u}$. Moreover, define the set

$$X = \{\boldsymbol{y} \in X : u(\boldsymbol{y}) = \overline{u}\}.$$

This set is open. In fact, for $\boldsymbol{x}_0 \in X$, consider $r > 0$ with $B_r(\boldsymbol{x}_0) \subset \Omega$. Then, applying Remark 3.57, there holds

$$\overline{u} = u(\boldsymbol{x}_0) = \fint_{B_r(\boldsymbol{x}_0)} u(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.$$

Note that this equality can only hold true if $u = \overline{u}$ on $B_r(\boldsymbol{x}_0)$. Therefore, $B_r(\boldsymbol{x}_0) \subset X$, and hence, X is open.

In addition, we have that $\Omega \cap \overline{X} \subseteq X$: It is clear that $\Omega \cap X \subseteq X$. Let us consider $\boldsymbol{y} \in (\Omega\cap\overline{X})\setminus(\Omega\cap X)$, and a sequence $\{\boldsymbol{y}_j\}_{j\geq 0} \subset \Omega\cap X$ that converges to $\boldsymbol{y}$. Then, since $\{\boldsymbol{y}_j\}_{j\geq 0} \subset \Omega \cap X$, we have that $u(\boldsymbol{y}_j) = \overline{u}$ for all $j \geq 0$. Moreover, due to the continuity of u, it follows that $u(\boldsymbol{y}) = u(\lim_{j\to\infty} \boldsymbol{y}_j) = \lim_{j\to\infty} u(\boldsymbol{y}_j) = \overline{u}$, i.e., $\boldsymbol{y} \in X$. Consequently, $\Omega \cap \overline{X} \subseteq X$, and, in fact, $\Omega \cap \overline{X} = X$.

Now, since $X \subset \Omega$ is open, $\Omega \cap \overline{X}$ is open also. This, however, is only possible if $X = \Omega$. □

Remark 3.62 Replacing u with $-u$ (which is harmonic if and only if u is harmonic) in the above Theorem 3.61, we see that all the '$\max$' can be exchanged with '$\min$' ('minimum principle').

One important application of maximum principles is the proof of uniqueness of a corresponding PDE problem (alternatively to, for example, energy methods). Let us consider the Poisson problem (2.1)–(2.2). Suppose that there are two solutions $u_1, u_2 \in C^2(\overline{\Omega})$, and let $w = u_1 - u_2$. Then, w is harmonic and $w = 0$ on $\partial\Omega$. Therefore, using Theorem 3.61, we see that

$$\max_{\boldsymbol{x}\in\overline{\Omega}} w(\boldsymbol{x}) = \max_{\boldsymbol{x}\in\partial\Omega} w(\boldsymbol{x}) = 0.$$

In the same way, recalling Remark 3.62, we have

$$\min_{\boldsymbol{x}\in\overline{\Omega}} w(\boldsymbol{x}) = \min_{\boldsymbol{x}\in\partial\Omega} w(\boldsymbol{x}) = 0.$$

Hence, $w \equiv 0$ on Ω, i.e., $u_1 \equiv u_2$.

3.5 Exercises

3.1. Prove (3.18).

3.2. Example 3.17 shows that the fundamental solution of the Laplace equation from Definition 3.27 satisfies (3.24) for $m = 2$. Prove that it also holds true for $m \geq 3$.

3.3. Given an open bounded domain $\Omega \subset \mathbb{R}^m$. Prove that

$$\int_\Omega |\Phi(x)| \,\mathrm{d}\boldsymbol{x}$$

exists and find an upper bound. Here, Φ is the fundamental solution of the Laplace equation from (3.24). Show that Φ is not integrable over $\mathbb{R}^m$.

3.4. Prove the symmetry of the Green function $G^{\boldsymbol{x}}$ from Definition 3.42; cf. Remark 3.45.

3.5. Find the Green function for the Poisson problem with Dirichlet boundary conditions in

  a) the quadrant $Q = (0, \infty)^2 \in \mathbb{R}^2$;

  b) the spherical shell $\{\boldsymbol{x} \in \mathbb{R}^m : \alpha < |\boldsymbol{x}| < \beta\}$, where $0 < \alpha < \beta$ are constants. What happens for $\beta \to \infty$?

3.6. In an open domain $\Omega \subset \mathbb{R}^m$ consider the Poisson problem with *Neumann* boundary conditions:

$$-\Delta u = f \qquad \text{in } \Omega \tag{3.63}$$

$$\frac{\partial u}{\partial \boldsymbol{n}} = g \qquad \text{on } \partial\Omega. \tag{3.64}$$

Here, $\frac{\partial u}{\partial \boldsymbol{n}} \equiv \nabla u \cdot \boldsymbol{n}$, where $\boldsymbol{n}$ is the unit outward vector on $\partial\Omega$.

  a) Show that a necessary condition for the above boundary value problem (3.63)–(3.64) to have a solution is

$$\int_\Omega f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} g(\boldsymbol{x}) \,\mathrm{d}s = 0.$$

  b) Show that two solutions of (3.63)–(3.64) differ by a constant. Hence, prove that imposing the additional condition $\int_{\partial\Omega} u(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = 0$ on the solution u guarantees its uniqueness.

  c) Let $G^{\boldsymbol{x}}(\boldsymbol{y}) = \Phi(\boldsymbol{x} - \boldsymbol{y}) - \varphi^{\boldsymbol{x}}(\boldsymbol{y})$ be the Green function for (3.63)–(3.64). Find a suitable PDE formulation (including boundary conditions) for the corrector function $\varphi^{\boldsymbol{x}}(\boldsymbol{y})$ and give a formula for the solution u in terms of $G^{\boldsymbol{x}}$ (and the data).

3.7. Prove Theorem 3.46 for the half space solution given in (3.50).

3.8. For $m \geq 3$ and sufficiently smooth functions f, g, consider the PDE problem

$$\begin{aligned} -\Delta u &= f \qquad \text{in } B_1(\mathbf{0}) \\ u &= g \qquad \text{on } \partial B_1(\mathbf{0}). \end{aligned}$$

Prove that there exists a constant $C > 0$, depending only on m, such that

$$\sup_{B_1(\mathbf{0})} |u| \leq C \left(\sup_{\partial B_1(\mathbf{0})} |g| + \sup_{B_1(\mathbf{0})} |f| \right).$$

3.9. Prove Remark 3.57.

3.10. Consider an open, bounded domain $\Omega \subset \mathbb{R}^m$, with smooth boundary $\partial\Omega$. Moreover, let $B_R(\boldsymbol{x}_0)$ be the smallest ball that contains Ω. Assume $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ solves the Poisson equation,

$$\begin{aligned} -\Delta u &= 1 \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega, \end{aligned}$$

where g is a smooth function.

a) Use the maximum principle to prove that

$$u(\boldsymbol{x}) \leq \frac{1}{2m} \left(R^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2 \right) + \sup_{\boldsymbol{x} \in \partial\Omega} g(\boldsymbol{x}).$$

Hint: Look at the function $v(\boldsymbol{x}) = u(\boldsymbol{x}) + \alpha|\boldsymbol{x} - \boldsymbol{x}_0|^2$ for a suitable constant $\alpha \in \mathbb{R}$.

b) Let $g \equiv 0$. Use a) to show that

$$\fint_{\Omega} |\nabla u(\boldsymbol{x})|^2 \, d\boldsymbol{x} \leq \frac{R^2}{2m}.$$